Senior Prize Exam
Spring 2014

1) Given a positive integer n, show that n^3+5n is divisible by 6.

2) Show that there are no four consecutive integers (i.e., integers of form n, $n+1$, $n+2$, $n+3$ for some n) each of which is a power with an integer exponent >1 of an integer.

3) Let a, b, c, d, p, and q be positive integers satisfying $ad-bc=1$ and $a/b>p/q>c/d$. Prove that $q\ge b+d$.[1]

4) Show that

$$\int_{-1}^{1}\ln(x+\sqrt{1+x^2})\,dx=0.$$

5) For a given integer $n\ge 2$, place n red points and n blue points in a row. A place between two adjacent points will be called an *even split* if cutting the row at that place will leave the same number of red and blue points to the left of the cut. Show that the number of color arrangements with exactly one even split is twice the number of color arrangements with no even split.

6) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\sum_{n=1}^{\infty}a_nb_n$ converges for every sequence $\{b_n\}_{n=1}^{\infty}$ satisfying $\sum_{n=1}^{\infty}|b_n|<+\infty$. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.

7) Let $P(x)$ be a polynomial of degree n for which $P(x)\ge 0$ for all real numbers x. Prove that

$$\sum_{k=0}^{n}P^{(k)}(x)\ge 0$$

for all real numbers x.

Soon after the exam, solutions will appear on the Web Site
http://www.sci.brooklyn.cuny.edu/~mate/prize/2014/

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

[1] By mistake, the our earlier formulation of the problem omitted d from the list of positive integers, and the text said: Let a, b, c, p, and q be positive integers satisfying $ad-bc=1$ and $a/b>p/q>c/d$. Prove that $q\ge b+d$.