All Problems on the Prize Exams
Spring 2015

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. The Junior Prize Exam was not given this year.

1) (SENIOR 1) Let p be a prime number, and let r be the remainder when p is divided by 30. Show that r is also prime or $r = 1$.

Source: Problem 81, Középiskolai Matematikai Lapok, Vol. IV, No. 11 (1952), p. 124. See

`http://db.komal.hu/scan/1952/11/95211124.g4.png`

Solution: We may assume that $p > 30$; otherwise, we would have $r = p$, and p is a prime. We have $r = p - 30q$ for some integer q; further $0 \le r < 30$. The prime divisors of 30 are 2, 3, and 5; since none of these is a divisor of p, they cannot be a divisor of r either. Thus, unless $r = 1$, the smallest prime divisor of r is ≥ 7. Unless $r = 1$ or r is this prime itself, we must have $r \ge 7^2 = 49$. Since $r < 30$, this is not possible; so, indeed, $r = 1$ or else is a prime.

2) (SENIOR 2) Given real numbers a, b, and c, show that

$$a^2 + b^2 + c^2 \ge ab + bc + ca.$$

Source: Problem 31, Középiskolai Matematikai és Fizikai Lapok, Vol. I, No. 3 (1925), p. 40. See

`http://db.komal.hu/scan/1925/04/92504040.g4.png`

Solution: We have

$$0 \le (a-b)^2 + (b-c)^2 + (c-a)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ca),$$

whence the assertion follows. Clearly, equality holds only in case $a = b = c$.

3) (SENIOR 3) Given six consecutive integers, show that there is one among them that is relatively prime to all others. (Two integers are called relatively prime if their greatest common divisor is 1.)

Source: Problem 2, first category, round 2 for 10th grades, Daniel Arany Mathematics Competition, 1960. See

Source:

`http://versenyvizsga.hu/external/vvszuro/vvszuro.php`

Solution: The difference of any two among six consecutive integers is at most 5. Since the greatest common divisor of two numbers also divides their difference, the only prime factors that the greatest common divisor of these two integers can have are 2, 3, and 5. So, in order to find a number among the six that is relatively prime to the others, we only need to find one number that is not divisible by 2, 3, and 5.

There is one among the six numbers, say n, that is divisible by 6. Then none of the numbers $n-1$, $n-5$, $n+1$, and $n+5$ is divisible by 2 or 3, and least two of them is among the six consecutive

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

numbers. Among these two, only one can be divisible by 5, since the difference only of $n+5$ and $n-5$ among them is divisible by 5, and not both of these numbers are among the six consecutive numbers. Thus, there will be one number among $n-1$, $n-5$, $n+1$, and $n+5$ that is not divisible by 5 and is among the six consecutive numbers. This number will be relatively prime to all the others.

4) (SENIOR 4) Let n be a positive integer, and let a_1, a_2, ..., a_n be real numbers. Write

$$f(x) = \sum_{k=1}^{n} a_k \sin kx.$$

Assume that $|f(x)| \le |x|$ for all $x > 0$. Prove that $|\sum_{k=1}^{n} k a_k| \le 1$.

Source: Based on Problem 4, APICS (Atlantic Provinces Council on the Sciences, Canada) Mathematics Contest 1993. See

`http://www.math.unb.ca/apics.papers/93/93.html`

Solution: We have

$$\lim_{x\to 0} \frac{\sin cx}{x} = c$$

for any real c. Hence

$$\lim_{x\to 0} \frac{f(x)}{x} = \sum_{k=1}^{n} k a_k.$$

In view of the inequality $|f(x)| \le |x|$, the absolute value of the right-hand side must be less than or equal to 1, establishing the result.

5) (SENIOR 5) Let $a_n \ge 0$ for all $n \ge 1$ and assume that

$$\frac{1}{n} \sum_{k=1}^{n} a_k \ge \sum_{k=n+1}^{2n} a_k$$

for $n \ge 1$. Show that $\sum_{k=1}^{\infty} a_k$ is convergent and its sum is less than $2ea$, where e is the base of the natural logarithm.

Source: Problem 6, Schweitzer Miklós Emlékverseny (Miklós Schweitzer Memorial Competition), Hungary 1958. See

`http://www.versenyvizsga.hu/`

Solution: We claim that we have

$$\sum_{k=1}^{2^n} a_k \le a_1 \prod_{k=0}^{n-1} \left(1 + 2^{-k}\right) \tag{1}$$

for all $n \ge 0$. This is easy to show by induction on n. Indeed, for $n = 0$ this just says $a_1 \le a_1$, since in this case the product on the right-hand side is empty, and the value of the empty product is 1 by convention. So let $n \ge 1$ and the assume that (1) is true for this n.

By the assumption of the problem, we have

$$2^{-n} \sum_{k=1}^{2^n} a_k \ge \sum_{k=2^n+1}^{2^{n+1}} a_k,$$

and so

$$(2) \qquad \left(1+2^{-n}\right)\sum_{k=1}^{2^n} a_k \ge \sum_{k=1}^{2^{n+1}} a_k.$$

By (1) and (2) we have

$$a_1 \prod_{k=1}^{n}\left(1+2^{-k}\right) = \left(1+2^{-n}\right)\ a_1 \prod_{k=1}^{n-1}\left(1+2^{-k}\right) \ge \left(1+2^{-n}\right)\ \sum_{k=1}^{2^n} a_k \ge \sum_{k=1}^{2^{n+1}} a_k;$$

the first inequality holds by (1), and the second one by (2). This completes the induction, establishing (1).

If $a_1 = 0$, the assertion of the problem immediately follows from (1), so assume that $a_1 > 0$. Writing $\log x$ for the natural logarithm of x and using the inequality $\log(1+x) < x$, valid for all $x > 0$, by (1) we have

$$\begin{aligned}
\log \sum_{k=1}^{2^{n+1}} a_k &\le \log a_1 + \sum_{k=0}^{n-1} \log\left(1+2^{-k}\right) \\
&< \log a_1 + \log 2 + \log\left(1+\frac{1}{2}\right) - \frac{1}{2} + \sum_{k=1}^{\infty} 2^{-k} \\
&= \log a_1 + \log 2 + \log\left(1+\frac{1}{2}\right) - \frac{1}{2} + 1
\end{aligned}$$

for $n > 2$, where the $-1/2$ in the second line cancels out the term of the sum for $k = 1$, which is not needed since the first two terms of the second sum in the first line are explicitly written out. Thus

$$\log \sum_{k=1}^{\infty} a_k \le \log a_1 + \log 2 + \log\left(1+\frac{1}{2}\right) - \frac{1}{2} + 1.$$

As $\log(1+1/2) \le 1/2$, the inequality

$$\sum_{k=1}^{\infty} a_k < 2ea_1.$$

follows.

6) (SENIOR 6) Let $n > 0$ be an integer. Consider a polynomial in n variables with real coefficients. We know that if every variable is ± 1, the value of the polynomial is positive or negative according as the number of variables having value -1 is even or odd. Prove that the degree of this polynomial is at least n.

Source: Problem 2, József Kürschák Mathematical Competition, Hungary, 1995. See

`http://www.artofproblemsolving.com/Forum/resources.php`

Click on Hungary under the heading National and Regional competitions.

Solution: We may assume that the polynomial is symmetric in its variables. In fact, if the polynomial is $R(x_1, x_2, \dots, x_n)$, we can take the polynomial

$$Q(x_1, x_2, \dots, x_n) = \sum_{\sigma} R(x_{\sigma(1)}, x_{\sigma(2)} \dots, x_{\sigma(n)}),$$

instead, where σ runs over all permutations of $\{1, 2, \dots, n\}$ (i.e., all one-to-one mappings of this set onto itself). We may also assume that for any i with $1 \le i \le n$, x_i occurs in each term of $Q(x_1, x_2, \dots, x_n)$ with exponent 0 or 1. Indeed, in every term we can replace x_i^l with 1 is l is even and with x_i if l is odd; this will not change the value of $Q(x_1, x_2, \dots, x_n)$ for $x_i = \pm 1$. In this way, the degree of $Q(x_1, x_2, \dots, x_n)$ will be at most n. We are going to show that there is a polynomial $P(x)$ such that

$$P\Big(\sum_{i=1}^{n} x_i\Big) = Q(x_1, x_2, \dots, x_n)$$

whenever $x_i = \pm 1$ for all i with $1 \le i \le n$, and, further, the degree of P is not higher than that of Q. Such a polynomial P can easily be constructed by recursion. Assume $P_0(x) = 0$ and $Q_0(x_1, x_2, \dots, x_n) = Q(x_1, x_2, \dots, x_n)$. At the kth step, we will eliminate all terms of degree $n-k$ from Q_k and introduce a corresponding term of degree $n-k$ into P, while preserving the symmetry of the polynomial Q_k.

So, assume that $P_k(x)$ and $Q_k(x_1, x_2, \dots, x_n)$ have already been constructed for some k with $0 \le k \le n$, and the degree of $Q_k(x_1, x_2, \dots, x_n)$ is at most $n-k$; and, further, the exponent of each occurrence of x_i in this polynomial for i with $1 \le i \le n$ is 0 or 1. Let c_k be the coefficient of the term $\prod_{i=1}^{n-k} x_i$ in this polynomial for some constant c_k (possibly $c_k = 0$), and let $P_{k+1}(x) = P_k(x) + c_k x^{n-k}$. Consider the polynomial

$$Q_k(x_1, x_2, \dots, x_n) - c_k \Big(\sum_{i=1}^{n} x_i\Big)^{n-k}.$$

In this polynomial, replace all occurrences of x_i^l with 1 if l is even, and with x_i if l is odd, and call the resulting polynomial $Q_{k+1}(x_1, x_2, \dots, x_n)$. The degree of this polynomial is at most $n-k-1$ (zero being the degree of a nonzero constant polynomial, and -1 being the degree of the identically zero polynomial), and

$$P_k\Big(\sum_{i=1}^{n} x_i\Big) + Q_k(x_1, x_2, \dots, x_n) = P_{k+1}\Big(\sum_{i=1}^{n} x_i\Big) + Q_{k+1}(x_1, x_2, \dots, x_n)$$

whenever $x_i = \pm 1$. For $k = n+1$ we will have $Q_{n+1}(x_1, x_2, \dots, x_n) = 0$ and

$$P_{n+1}\Big(\sum_{i=1}^{n} x_i\Big) = Q(x_1, x_2, \dots, x_n)$$

whenever $x_i = \pm 1$ $(1 \le i \le n)$. For k with $0 \le k \le n$, taking $x_i = -1$ for $i \le k$ and $x_i = 1$ for $i > k$, we therefore have $P(n-2k) = (-1)^k$. This means that $P(x)$ has at least n sign changes, and so it has at least n (real) zeros. Therefore, the degree of $P(x)$ is at least n, completing the proof.

7) (SENIOR 7) Prove that the equation $y' = y^2 + x$, $y(0) = 0$ does not have a solution on the interval $(0, 3)$.

Source: Ural State University- DMM Olympiad, 2005, Problem 13.

`http://www.artofproblemsolving.com/Forum/resources.php`

Click on the last item in the middle column, with the heading Undergraduate Competitions.

Solution: Assuming $y(x)$ is such a solution, we have

$$y(x) = \int_0^x \left(\big(y(t)\big)^2 + t \right) dt \geq \int_0^x t\, dt = \frac{x^2}{2} \geq 0.$$

On the other hand,

$$\frac{y'}{y^2 + x} = 1,$$

and so

$$\frac{y'}{y^2 + 1} \geq 1 \qquad \text{for} \quad x \geq 1.$$

Integrating this on the interval $[1, x]$ for $x > 1$ we obtain

$$\begin{aligned} \int_1^x \frac{y'(t)}{\big(y(t)\big)^2 + 1}\, dt &= \int_{y(1)}^{y(x)} \frac{dy}{y^2 + 1} = \arctan y(x) - \arctan y(1) \\ &\geq \int_1^x 1\, dt = x - 1. \end{aligned}$$

Noting that $y(1) \geq 0$ and so $\arctan y(1) \geq 0$ and $\arctan y(x) \leq \lim_{y \to \infty} \arctan y = \pi/2$, this inequality implies that $x - 1 \leq \pi/2$, and so $x \leq 1 + \pi/2 < 3$. Hence there is no function $y(x)$ with $y(0) = 0$ that is continuous at 0 and satisfies the differential equation $y' = y^2 + x$ on the interval $(0, 3)$.