Senior Prize Exam
Spring 2015

1) Let p be a prime number, and let r be the remainder when p is divided by 30. Show that r is also prime or $r = 1$.

2) Given real numbers a, b, and c, show that

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

3) Given six consecutive integers, show that there is one among them that is relatively prime to all others. (Two integers are called relatively prime if their greatest common divisor is 1.)

4) Let n be a positive integer, and let a_1, a_2, ..., a_n be real numbers. Write

$$f(x) = \sum_{k=1}^{n} a_k \sin kx.$$

Assume that $|f(x)| \leq |x|$ for all $x > 0$. Prove that $|\sum_{k=1}^{n} ka_k| \leq 1$.

5) Let $a_n \geq 0$ for all $n \geq 1$ and assume that

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \sum_{k=n+1}^{2n} a_k$$

for $n \geq 1$. Show that $\sum_{k=1}^{\infty} a_k$ is convergent and its sum is less than $2ea$, where e is the base of the natural logarithm.

6) Let $n > 0$ be an integer. Consider a polynomial in n variables with real coefficients. We know that if every variable is ± 1, the value of the polynomial is positive or negative according as the number of variables having value -1 is even or odd. Prove that the degree of this polynomial is at least n.

7) Prove that the equation $y' = y^2 + x$, $y(0) = 0$ does not have a solution on the interval $(0, 3)$.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2015/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.