All Problems on the Prize Exams
Spring 2016

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Let n be a positive integer. Show that

$$30^{10n+1} + 5^{21n}$$

is divisible by 31.

Source: Problem 1975, Középiskolai Matematikai Lapok, Vol. 18/2, October 1910, p. 3
`http://db.komal.hu/scan/1910/10/91010025.g4.png`

Solution: The proof of the assertion can perhaps be described in the language of congruences the simplest. Given integers a, b, and c, we say that a is congruent to b modulo c, in notation

$$a \equiv b \mod c,$$

if $b - a$ is a multiple of c (or, in other words, $b - a$ is divisible by c). Here some authors assume that $c \geq 2$, though this assumption is unnecessary.[1]

For any integers $m > 0$, a, b, and c we have

$$(ac + b)^m \equiv b^m \mod c.$$

This easily follows from the basic properties of congruences, but even without using these properties, one can see this directly by noting that in the binomial expansion of $(ac + b)^m$ every term except b^m contains c as a factor. Using this, we have

$$\begin{aligned} 30^{10n+1} + 5^{21n} &= (31-1)^{10n+1} + (5^3)^{7n} = (31-1)^{10n+1} + (4 \cdot 31 + 1)^{7n} \\ &\equiv (-1)^{10n+1} + 1^{7n} \equiv -1 + 1 \equiv 0 \mod 31, \end{aligned}$$

establishing the assertion.[2]

2) (JUNIOR 2 and SENIOR 2) Write α, β, and γ for the roots of the equation

$$x^3 - 5x^2 - 9x + 45 = 0.$$

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{AMS}$-TEX was used for typesetting.

[1] The case of negative c is not interesting, since $a \equiv b \mod c$ means the same as $a \equiv b \mod (-c)$. The case $c = 1$ is not interesting, since $a \equiv b \mod 1$ is true for any integers a and b. Finally, the case $c = 0$ is not interesting, since $a \equiv b \mod 0$ means the same as $a = b$. Nevertheless, statements of a number of results about congruences becomes unnecessarily more complicated if one requires that $c \geq 2$.

[2] Note that in the last displayed line, the last two $\equiv$ symbols could be replaced with $=$; however, we did not want to surround the symbol $\equiv$ with equality symbols. So, after the first use of the symbol $\equiv$, we continued to use the symbol $\equiv$.

Given that we know that $\beta = -\alpha$, find the roots of the equations.

Source: Problem 1434, Középiskolai Matematikai Lapok, Vol. XIII.3, November 1905, p. 58. See

`http://db.komal.hu/scan/1905/11/90511058.g4.png`

Solution: We have

$$x^3 - 5x^2 - 9x + 45 = 0 = (x-\alpha)(x-\beta)(x-\gamma) = (x-\alpha)(x+\alpha)(x-\gamma)$$
$$= (x^2-\alpha^2)(x-\gamma) = x^3 - \gamma x^2 - \alpha^2 x + \alpha^2\gamma.$$

Equating the coefficients on the sides, this gives $\alpha = \pm 3$ and $\gamma = 5$. The choice $\alpha = 3$ and $\alpha = -3$ leads to the same result (i.e., that both 3 and -3 are roots of the equation). That is, the roots of the equation are 3, -3, and 5.

3) (JUNIOR 3 and SENIOR 3) When is the sum of the cubes of three consecutive integers (i.e., integers that are adjacent, or following one another) divisible by 18?

Source: Problem 1, second category, round 1 for grades 11–12, Daniel Arany Mathematics Competition, 1949. See

`http://versenyvizsga.hu/external/vvszuro/vvszuro.php`

Solution: Let the integers be $n-1$, n, and $n+1$. Then

$$(n-1)^3 + n^3 + (n+1)^3 = 3n^3 + 6n = 3n(n^2+2).$$

In order for this to be divisible by 18, we need to make sure that $n(n^2+2)$ is divisible by 6. Since n and n^2+2 have the same parity, $n(n^2+2)$ is divisible by 2 if and only if n is even.

On the other hand, $n(n^2+2)$ is always divisible by 3. Indeed, this is certainly the case if n is divisible by 3. Assume this is not the case. Then $n = 3k \pm 1$ for some k, and so

$$n^2 + 2 = (9k^2 \pm 6k + 1) + 2 = 9k^2 \pm 6k + 3,$$

which is divisible by 3. So

$$(n-1)^3 + n^3 + (n+1)^3$$

is divisible by 18 if and only if n is even. In other words, the sum of the cubes of three consecutive integers is divisible by 18 if and only if the middle number is even.

Note: To show that n^2+2 is divisible by 3 in case n is not divisible by 3, one can also argue that $n^2 + 2 = (n^2-1) + 3$ is divisible by 3 according to the following

Theorem (FERMAT). *Let p be a prime and let n be and integer not divisible by p. Then $n^{p-1} - 1$ is divisible by p.*

4) (JUNIOR 4) At a gathering, call a person an outsider if he or she knows at most three other persons at the gathering (knowing a person is mutual, that is, if A knows B then B also knows A). Show that if each person knows at least three outsiders, then everybody is an outsider.

Source: Problem Gy.3214 Középiskolai Matematikai és Fizikai Lapok, Vol. 48/6., September 1998, p. 360. See

`http://db.komal.hu/scan/1998/09/MAT9806.PS`

Solution: Call a person an insider if he or she is not an outsider. Since each outsider knows at most three other persons, and he or she knows at least three outsiders, he or she knows only these

three outsiders, and so he or she does not know any insiders. Since knowing a person is mutual, an insider does not know any outsider. So, if there were an insider at the gathering, this would contradict the assumption that every person knows at least three outsiders.

5) (JUNIOR 5) Given a convex quadrilateral such that its two diagonals divide it into four triangles of the same area. Prove that the quadrilateral is a parallelogram.

Source: Problem 7, first category, round 1 for 9th grades, Daniel Arany Mathematics Competition, 1971. See

`http://versenyvizsga.hu/external/vvszuro/vvszuro.php`

Solution: Given a convex quadrilateral $ABCD$, let E be the intersection of its two diagonals AC and BD. As the areas of the triangles AEB and ADE are equal, we must have $EB = ED$, since the altitudes dropped from the vertex A of these two triangles are the same. Similarly, $AE = EC$. Hence the triangles AEB and CED are congruent, since their angles at E are equal, and the corresponding two sides adjacent to E are also equal. Therefore $AB = CD$. The angles ABE and CDE are also equal, so $AB \parallel CD$. This shows that the quadrilateral $ABCD$ is indeed a parallelogram.

6) (JUNIOR 6) Let p and q be prime numbers with $p > q > 3$. Show that $p^2 - q^2$ is divisible by 24.

Source: Problem 1027, Középiskolai Matematikai és Fizikai Lapok, Vol. XII No. 3, p. 63 (corrected minor error). See

`http://db.komal.hu/scan/1935/11/93511064.g4.png`

Solution: We have $p^2 - q^2 = (p - q)(p + q)$. Both $p - q$ and $p + q$ are even, given that p and q are both odd. Furthermore $(p + q) - (p - q) = 2q$ is not divisible by 4, so one of $p + q$ and $p - q$ must be divisible by 4. Hence, $(p - q)(p + q)$ is divisible by 8.

On the other hand, $p^2 - 1$ is divisible by 3. Indeed, one of the numbers $p - 1$, p, and $p + 1$ must be divisible by 3. As $p > 3$ is a prime, p itself is not divisible by 3. Hence, $p^2 - 1 = (p-1)(p+1)$ is divisible by 3. Similarly, $q^2 - 1$ is divisible by 3. Therefore $p^2 - q^2 = (p^2 - 1) - (q^2 - 1)$ is divisible by 3. Therefore $p^2 - q^2$ is divisible by $8 \cdot 3 = 24$.

7) (JUNIOR 7) Show that from any given 7 integers one can select 4 whose sum is divisible by 4.

Source: Problem 3, category 3, 2nd round, grades 11-12, Országos Középiskolai Tanulmányi Verseny matematikából (Hungarian National Scholarly Competition in Mathematics for Secondary Schools), 1967. See

`http://www.versenyvizsga.hu/`

Solution: The seven given numbers can be replaced by their residue classes modulo 4. Then we need to select, with repetitions, seven of the four residue classes modulo 4. There are $\binom{4+7-1}{7} = \binom{10}{7} = \binom{10}{3} = 120$ ways to select these residue classes, and so in principle it would be possible to test all possible selections. The practical question is how to cut down on the number of selections that need to be tested.

Write $[x]$ for the residue class of the integer x modulo 4. A list of at most seven residue classes can be represented by a four-tuple (x_0, x_1, x_2, x_3) with $x_0 + x_1 + x_2 + x_3 \leq 7$, where x_i represents the number of times the residue class $[i]$ occurs on the list. Call such a four-tuple *good* if one can select four of these residue classes whose sum is $[0]$. Two observations are in order. If the four-tuple (x_0, x_1, x_2, x_3) is good, then any of its cyclic permutations is also good. In fact, if we can select the residue classes y_1, y_2, y_3, y_4 such that $y_1 + y_2 + y_3 + y_4 = [0]$ from the residue-classes represented by this four-tuple, then the four-tuple (x_1, x_2, x_3, x_0) represents number of times the residue classes $[0] = [1] - [1]$, $[1] = [2] - [1]$, $[2] = [3] - [1]$, and $[3] = [0] - [1]$ occur on the corresponding list, and

from this list we can select the residue classes $y_1 - [1]$, $y_2 - [1]$, $y_3 - [1]$, $y_4 - [1]$, whose sum is [0]. We can also reverse the above four-tuple; indeed, the four-tuple (x_3, x_2, x_1, x_0) represents number of times the residue classes $[0] = [3] - [3]$, $[1] = [3] - [2]$, $[2] = [3] - [1]$, and $[3] = [3] - [0]$ occur on the corresponding list, and from this list we can select the residue classes $[3] - y_1$, $[3] - y_2$, $[3] - y_3$, and $[3] - y_4$, whose sum is [0].

We need to show that every four-tuple (x_0, x_1, x_2, x_3) with with $x_0 + x_1 + x_2 + x_3 = 7$ is good. In order to do this, a simple collection of good four-tuples will help: $(4, 0, 0, 0)$ is good, since $[0] + [0] + [0] + [0] = [0]$; $(2, 1, 0, 1)$ is good, since $[0] + [0] + [1] + [3] = [0]$; finally, $(2, 0, 2, 0)$ is good, since $[0]+[0]+[2]+[2] = [0]$. When considering a four-tuple (x_0, x_1, x_2, x_3) with $x_0+x_1+x_2+x_3 = 7$, we say that it avoids the pattern (z_0, z_1, z_2, z_3) if at least one of the inequalities $x_i \geq z_i$ $(0 \leq i \leq 3)$ fails. In order for a four-tuple not to be good, it must avoid the three listed good patterns, as well as all the patterns resulting from them by cyclic permutations or reversals.

Assume, on the contrary to the assertion of a problem, that a four-tuple (x_0, x_1, x_2, x_3) with $x_0+x_1+x_2+x_3 = 7$ is not good. Then there is no x_i with $x_i \geq 4$, since we need to avoid any cyclic permutation of the pattern $(4, 0, 0, 0)$. Hence, there must be an x_i that is 2 or 3. We may assume that $i = 0$; that is, x_0 is 2 or 3. In order to avoid the pattern $(2, 1, 0, 1)$, we must have $x_1 = 0$ or $x_3 = 0$. We may assume that $x_1 = 0$, since the four-tuple (x_0, x_1, x_2, x_3) and (x_0, x_3, x_2, x_1) can be transformed into each other by a reversal and a cyclic permutation (from (x_0, x_1, x_2, x_3) we get (x_3, x_2, x_1, x_0) by reversal, and then a cyclic permutation gives (x_0, x_3, x_2, x_1)). In order to avoid the pattern $(2, 0, 2, 0)$, we must have $x_2 = 0$ or $x_2 = 1$. In case $x_2 = 0$ we must have $x_3 \geq 4$ (since $x_0 \leq 3$ and $x_1 = 0$), so a cyclic permutation of the pattern $(4, 0, 0, 0)$ is not avoided. In case $x_2 = 1$, we must have $x_3 \geq 3$, and the pattern $(1, 0, 1, 2)$ is not avoided; since this pattern is a cyclic permutation of the pattern $(2, 1, 0, 1)$, this shows that not all good patterns can be avoided. This completes the proof.

8) (SENIOR 4) Find positive integers x, y, and z such that $x < y < z$ and x^2, y^2, and z^2 form an arithmetic progression, and for which y is the least possible.

Source: Problem 19, Középiskolai Matematikai Lapok, Vol. 1., 1894–95, p. 24. See

`http://db.komal.hu/scan/1894/00/89400024.g4.png`

Solution: Write $u = z - y$ and $v = y - x$. Then we have

$$z^2 - y^2 = (y + u)^2 - y^2 = u^2 + 2uy \quad \text{and}$$
$$y^2 - x^2 = y^2 - (y - v)^2 = -v^2 + 2vy,$$

and so the equation $z^2 - y^2 = y^2 - x^2$ becomes

$$y = \frac{u^2 + v^2}{2(v - u)}. \tag{1}$$

If t is a common divisor of u and v, say $u = t\bar{u}$, $v = t\bar{v}$, then this equation says that $y = t\bar{y}$ with

$$\bar{y} = \frac{\bar{u}^2 + \bar{v}^2}{2(\bar{v} - \bar{u})}.$$

Then with $\bar{z} = \bar{y}+\bar{u}$ and $\bar{x} = \bar{y}-\bar{v}$, and the triple $\bar{x}^2$, $\bar{y}^2$, and $\bar{z}^2$ also form an arithmetic progression; further, we also have $x = t\bar{x}$, $t\bar{y}$ and $z = t\bar{z}$. So, to obtain the triple x, y, z with y the least possible, u and v must be relatively prime.

Equation (1) means that $v-u$ is a divisor of u^2+v^2. As

$$2u^2 = (u^2+v^2)+(u-v)(u+v) \quad \text{and}$$
$$2v^2 = (u^2+v^2)-(u-v)(u+v),$$

this means that $u-v$ is a divisor of both $2u^2$ and and $2v^2$. The assumption that u and v are relatively prime means that $v-u=1$ or $v-u=2$. The former would mean that u^2+v^2 is odd, which is not possible according to equation (1); that is, $v=u+2$. Then equation (1) becomes

$$2y = u^2+2u+2.$$

This means that u must be even. The choice $u=2$ gives the least possible value of y. We then have $y=5$, $v=u+2=4$, $x=y-v=1$ and $z=y+2=7$. This gives the triple $x=1$, $y=5$, and $x=7$ as the triple satisfying the requirements.

9) (SENIOR 5) Given an arbitrary positive integer n, show that

$$\sqrt{1+\sqrt{2+\sqrt{3+\ldots+\sqrt{n}}}} < 2.$$

Source: Based on Problem 2, third category, round 2 for 10th grades, Daniel Arany Mathematics Competition, 1978. See

`http://versenyvizsga.hu/external/vvszuro/vvszuro.php`

The Problem cited makes the weaker assertion that the left-hand side above is less than 4.

Solution: Let $x_{n+1}=0$, and for k with $1 \le k \le n$ let

$$x_k = \sqrt{k+x_{k+1}}.$$

Then x_1 is equal to the expression on the left-hand side of the inequality asserted by the problem. We are first going to show that $x_k \le k$ for all integers k with $3 \le k \le n+1$. This is certainly true for $k=n+1$. Let $3 \le k \le n$, and assume that $x_{k+1} < k+1$. Then we have

$$x_k = \sqrt{k+x_{k+1}} < \sqrt{k+(k+1)} = \sqrt{2k+1} < k,$$

where the last inequality is equivalent to $k^2 > 2k+1$, i.e., to $(k-1)^2 > 2$, which is certainly true if $k \ge 3$.

Hence $x_3 < 3$ and so

$$x_2 = \sqrt{2+x_3} < \sqrt{2+3} = \sqrt{5} < 3,$$

and so

$$x_1 = \sqrt{1+x_2} < \sqrt{1+3} = \sqrt{4} = 2,$$

which is what we wanted to prove.

10) (SENIOR 6) For every positive integer n let $f(n) > 0$, and assume that for all positive integers m and n we have $f(m+n) \le f(m)+f(n)$. Prove that the limit

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists and equals

$$\inf_{n>0} \frac{f(n)}{n}.$$

Source: Problem 5, Part II, Stanford University Mathematics Ph.D. Qualifying Exam in Real Analysis, September 2006. See

`http://mathematics.stanford.edu/academics/graduate/phd-program/phd-qualifying-exams/past-qualifying-exams/`

Solution: Let

$$A = \inf_{n>0} \frac{f(n)}{n}. \tag{1}$$

Then $0 \le A \le f(1)/1$, so A is finite. Let $\epsilon > 0$ be arbitrary, and let $m > 0$ be such that

$$\frac{f(m)}{m} < A + \epsilon,$$

and let

$$B = \max\{f(j) : 1 \le j \le m\}$$

(we need to allow equality on the right in case $m = 1$). Let $n > 0$ be arbitrary, and let k be an integer such that $mk \le n < m(k+1)$. Then

$$f(n) = f\big(mk + (n - mk)\big) \le kf(m) + f(n - mk) < km(A + \epsilon) + B.$$

Thus

$$\frac{f(n)}{n} < \frac{km}{n}(A + \epsilon) + \frac{B}{n} \le (A + \epsilon) + \frac{B}{n}.$$

Making $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{f(n)}{n} \le A + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this together with (1) implies that

$$\lim_{n\to\infty} \frac{f(n)}{n} = A.$$

11) (SENIOR 7) Find a noncommutative group G such that for all $x \in G$ we have $x^3 = e$, where e denotes the identity element of G.

Source: David Finston, oral communication.

Solution: Such a group is formed by the *upper triangular unit matrices* of size 3×3 over a field of *characteristic* 3.

Let n be a positive integer (for the present problem we are only interested in the case $n = 3$, but the case of an arbitrary n is simple enough to discuss). An $n \times n$ matrix (a_{ij}) is called an *upper triangular matrix* if $a_{ij} = 0$ whenever $1 \le j < i \le n$. Such a matrix is called an *upper triangular unit matrix* if we also have $a_{ii} = 1$ for all i with $1 \le i \le n$ (here 0 and 1 denote the zero element and the multiplicative unit element of the field over which these matrices are considered). The

product $C = (c_{ij})$ of two such matrices $A = (a_{ij})$ and $B = (b_{ij})$ is another upper triangular unit matrix. Indeed,

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

In each term on the right-hand side, one of the factors is zero unless $i \le j$ and $j \le k$, and so the sum is zero unless $i \le k$. If $i = k$, the only way to get a nonzero sum if $i = j = k$, and in this case the sum is 1.

The inverse of an upper triangular unit matrix of size $n \times n$ is again an upper triangular unit matrix. Indeed, if $A = (a_{ij})$ is such a matrix, then its inverse $X = (x_{ij})$ can be found by solving the equations

$$\sum_{j=1}^{n} a_{ij} x_{jk} = \delta_{ik},$$

where δ_{ik} is Kronecker's delta, i.e.,

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \ne k, \end{cases}$$

and $1 \le i, k \le n$. We have $a_{ii} = 1$ and $a_{ij} = 0$ if $j < i$, since A is and upper triangular unit matrix. Stipulating that X is also an upper triangular unit matrix, i.e., that $x_{ii} = 1$ and $x_{ij} = 0$ if $j < i$, the above equations are satisfied if $k \le i$, and for $k > i$ the equations can be written as

$$x_{ik} = -\sum_{j=i+1}^{k} a_{ij} x_{jk}.$$

For each k with $1 \le k \le n$ these equations are solvable simply by determining x_{ik} for $i = k - 1$, $k - 2$, …, 1 (in this order) by evaluating the right-hand side.

The simplest field of characteristic 3 is the Galois field $\mathbb{F}_3$. The elements of this are the integers modulo 3, with addition and multiplication taken modulo 3. In such a field, $3x = x + x + x = 0$ (this is the equation that is meant by saying that the field has *characteristic* 3).

Now, let F be a field of characteristic 3, and let M be the field group of upper triangular unit matrices of size 3×3 over F. Then the multiplication in M is not commutative. Indeed, with

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

we have

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $A \in M$, the characteristic polynomial of A is $(x - 1)^3$. By the Cayley-Hamilton theorem, A is a zero of its own characteristic polynomial. That is, writing I for the unit matrix over F, we have[3]

$$0 = (A - I)^3 = A^3 - 3A^2 + 3A - I = A^3 - I,$$

[3]One needs to be a little cautious here. We are going to apply the Binomial Theorem for matrices. The proof of this theorem makes use of the commutativity of multiplication; so $(P + Q)^n$ can be expressed by the Binomial Theorem only if the matrices P and Q commute. In the present case, we will use it with $P = A$ and $Q = I$, and A and I clearly commute.

where the last equation holds since F has characteristic 3. This equation shows that for every $A \in M$ we have $A^3 = I$, which is what we wanted to show.

Note: For any prime $p \geq 3$, a similar construction with $p \times p$ matrices shows that there is a noncommutative group G such that $x^p = e$ for any $x \in G$. The noncommutativity of the multiplicative group of upper triangular $p \times p$ unit matrices is shown by the two matrices which have the 3×3 matrices A or B in the top left corners, all 1s in the main diagonal, and 0s everywhere else.

For $p = 2$ this cannot be done. Indeed, if G is a group such that $x^2 = e$ for every $x \in G$, then we have $x^{-1} = x$ for all $x \in G$, and so given, $a, b \in G$, we have

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba,$$

so such a group is necessarily commutative.