

JUNIOR PRIZE EXAM  
SPRING 2016

1) Let  $n$  be a positive integer. Show that

$$30^{10n+1} + 5^{21n}$$

is divisible by 31.

2) Write  $\alpha$ ,  $\beta$ , and  $\gamma$  for the roots of the equation

$$x^3 - 5x^2 - 9x + 45 = 0.$$

Given that we know that  $\beta = -\alpha$ , find the roots of the equations.

3) When is the sum of the cubes of three consecutive integers (i.e., integers that are adjacent, or following one another) divisible by 18?

4) At a gathering, call a person an outsider if he or she knows at most three other persons at the gathering (knowing a person is mutual, that is, if  $A$  knows  $B$  then  $B$  also knows  $A$ ). Show that if each person knows at least three outsiders, then everybody is an outsider.

5) Given a convex quadrilateral such that its two diagonals divide it into four triangles of the same area. Prove that the quadrilateral is a parallelogram.

6) Let  $p$  and  $q$  be prime numbers with  $p > q > 3$ . Show that  $p^2 - q^2$  is divisible by 24.

7) Show that from any given 7 integers one can select 4 whose sum is divisible by 4.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE

<http://www.sci.brooklyn.cuny.edu/~mate/prize/2016/>