Senior Prize Exam
Spring 2016

1) Let n be a positive integer. Show that

$$30^{10n+1} + 5^{21n}$$

is divisible by 31.

2) Write α, β, and γ for the roots of the equation

$$x^3 - 5x^2 - 9x + 45 = 0.$$

Given that we know that $\beta = -\alpha$, find the roots of the equations.

3) When is the sum of the cubes of three consecutive integers (i.e., integers that are adjacent, or following one another) divisible by 18?

4) Find positive integers x, y, and z such that $x < y < z$ and x^2, y^2, and z^2 form an arithmetic progression, and for which y is the least possible.

5) Given an arbitrary positive integer n, show that

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + \ldots + \sqrt{n}}}} < 2.$$

6) For every positive integer n let $f(n) > 0$, and assume that for all positive integers m and n we have $f(m+n) \le f(m) + f(n)$. Prove that the limit

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists and equals

$$\inf_{n>0} \frac{f(n)}{n}.$$

7) Find a noncommutative group G such that for all $x \in G$ we have $x^3 = e$, where e denotes the identity element of G.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2016/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.