All Problems on the Prize Exams
Spring 2017

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. The Junior Prize Exam was not given this year.

1) (SENIOR 1) Show that for any integer n, the number $n^3 + 11n$ is divisible by 6.

Source: Problem 918, Középiskolai Matematikai Lapok, Vol. X1/1, September 15, 1934.

`http://db.komal.hu/scan/1934/09/93409030.g4.png`

Solution: We need to show that $n^3 + 11n$ is divisible both by 2 and 3. We have

$$n^3 + 11n = n^2(n + 11).$$

This is clearly divisible by 2, since for even n the first factor is even, and for odd n the second factor is so.

As for divisibility by 3, it is enough to show that

$$(n^3 + 11n) - 3 \cdot 4n = n^3 - n = (n^2 - 1)n = (n - 1)n(n + 1)$$

is divisible by 3. This is certainly true, since one of the factors on the right-hand side is divisible by 3.

2) (SENIOR 2) In a class of 34 students, there are 17 females and 17 males. Assume they are all sitting at a round table. Show that there is at least one (male or female) student with two female neighbors.

Source: Problem 2 Dániel Arany Mathematical Competition, round I, beginners, category I-II, academic year 2013/2014; on p. 2 of the pdf file at the website, the academic year is misstated as 2012/1013, but the main page states it correctly. See

`http://www.bolyai.hu/aranydaniel.htm`

Solution: Assume there are no such students. Divide the students into maximal groups of students of the same sex sitting next to one another. No such group of females can contain more than two members, since otherwise there would be a female student with two female neighbors. This means that there must be at least 9 such groups of females. As for the group of males, there can be no group that contains only a single member, since then the member of that group would have two female neighbors. This means that there must be at most 8 group (since 17 is an odd number, at least one group has to have more than two members). However, the number of female groups must be equal to the number of male groups, a contradiction. This shows that there indeed must be a student with two female neighbors.

3) (SENIOR 3) Let p and q be positive integers, and assume that all solutions of the equations

$$x^2 + px - q = 0 \qquad \text{and} \qquad x^2 + px + q = 0$$

are integers. Show that there are nonzero integers a and b such that $p^2 = a^2 + b^2$.

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

Source: Based on Problem 10, Euclid Contest (Grade 12), 1998, Canadian Mathematics Competition, University of Waterloo, Waterloo, Ontario, Canada. See

`http://www.cemc.uwaterloo.ca/contests/past_contests.html`

Solution: The discriminants of both equations must be squares of integers; therefore, there are integers u and v such that $p^2 + 4q = u^2$ and $p^2 - 4q = v^2$. Then

$$2p^2 = u^2 + v^2,$$

and so

$$p^2 = \left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2.$$

Furthermore,

$$8q = u^2 - v^2,$$

and so u and v must have the same parity,[1] so $(u+v)/2$ and $(u-v)/2$ are integers. The last equation also show that neither of these integers is zero. Thus, $p^2 = a^2 + b^2$ with the nonzero integers $a = (u+v)/2$ and $b = (u-v)/2$.

4) (SENIOR 4) Show that

$$\sin\frac{\pi}{10}\ \sin\frac{3\pi}{10} = \frac{1}{4}.$$

Source: Based on Problem 830, Középiskolai Matematikai Lapok, Vol. VII/10, June 1900, p. 168:

`http://db.komal.hu/scan/1900/06/90006168.g4.png`

Solution: Using the identity

$$2\sin x\ \sin y = \cos(x-y) - \cos(x+y),$$

we need to show that

$$\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}. \tag{1}$$

Writing

$$\zeta = e^{i\pi/5} = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5},$$

we have

$$\cos\frac{\pi}{5} = \frac{\zeta + \zeta^{-1}}{2}$$

and

$$\cos\frac{2\pi}{5} = \frac{\zeta^2 + \zeta^{-2}}{2}.$$

That is, we need to show that

$$-\zeta^2 + \zeta + \zeta^{-1} - \zeta^{-2} = 1. \tag{2}$$

[1] I.e., they are either both even or they are both odd.

To show this, we need some basic equations involving ζ. To start with, we have

$$\zeta^5 = e^{i\pi} = -1. \tag{3}$$

Therefore

$$0 = \zeta^5 + 1 = (\zeta + 1)(\zeta^4 - \zeta^3 + \zeta^2 - \zeta + 1).$$

Noting that the first factor on the right-hand side is not zero since $\zeta \neq -1$, the second factor must be zero; that is

$$-\zeta^4 + \zeta^3 - \zeta^2 + \zeta = 1. \tag{4}$$

Using equations (3) and (4), the left-hand side of (2) equals

$$-\zeta^2 + \zeta + \zeta^{-1} - \zeta^{-2} = \zeta^2 - \zeta - \zeta^5(\zeta^{-1} - \zeta^{-2}) = -\zeta^4 + \zeta^3 - \zeta^2 + \zeta = 1;$$

the first equation holds in view of (3), and the third one in view of (4). This establishes (2).

Note: The above proof using complex roots of unity can easily be formulated geometrically involving a regular decagon (10-gon), but the proof as presented is easier to read. We will describe such a geometric proof.

Given a circle of radius 1 and center O, let P_0, P_1, $\ldots$, P_9 be the vertices of the inscribed decagon; it is natural to make O to be the center of the coordinate system, and to put the point P_0 on the positive x-axis; that is P_0 is the point with coordinates $(1, 0)$, and then label the points counterclockwise. Let P'_k be the projection of P_k onto the line OP_0. Equation (1) is equivalent to saying that $\overline{P'_2P'_1} = 1/2$.

To see this, first note that $OP_2 \parallel P_8P_1$. This can be seen by a simple calculation of angles: We have $\angle P_8OP_1 = 3\pi/5$, and so, noting that the triangle P_1OP_8 is isosceles, we have

$$\angle OP_1P_8 = \frac{\pi - \angle P_8OP_1}{2} = \frac{\pi}{5}.$$

Writing Q for the intersection of P_8P_1 and OP_0, we have

$$\angle P_0QP_1 = \angle P_0OP_1 + OP_1P_8 = \frac{\pi}{5} + \frac{\pi}{5} = \frac{2\pi}{5} = \angle P_0OP_2,$$

showing that indeed $OP_2 \parallel P_8P_1$. A similar argument shows that $P_8P_1 \parallel P_9P_0$. Indeed,

$$\angle OP_0P_9 = \frac{\pi - \angle P_0OP_9}{2} = \frac{\pi - \pi/5}{2} = \frac{2\pi}{5} = \angle P_0OP_2.$$

Hence, noting that $\overline{P'_2P_2} = \overline{P'_2P_8}$ and $\overline{P'_1P_1} = \overline{P'_1P_9}$, we can see that $\triangle OP'_2P_2 \simeq \triangle QP_2P_8$ and $\triangle QP'_1P_1 \simeq \triangle P_0P'_1P_9$. Thus

$$\overline{P'_2P'_1} = \overline{P'_2Q} + \overline{QP'_1} = \overline{OP'_2} + \overline{P'_1P_0}.$$

Since

$$1 = \overline{OP'_2} + \overline{P'_1P_0} + \overline{P'_2P'_1},$$

it follows that

$$\overline{P_2' P_1'} = \frac{1}{2},$$

as we wanted to show.

5) (SENIOR 5) Calculate the integral

$$\int_{-\pi}^{\pi} \frac{x^2}{1 + \sin x + \sqrt{1 + \sin^2 x}}\, dx.$$

Source: Problem 6, Stanford Math Tournament 2011, Calculus.
`https://sumo.stanford.edu/smt/`

Solution: Write $f(x)$ for the integrand. We have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\, dx &= \int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \\ &= \int_{0}^{\pi} f(-x)\, dx + \int_{0}^{\pi} f(x)\, dx = \int_{0}^{\pi} \big(f(-x) + f(x)\big)\, dx, \end{aligned}$$

where the second equality is obtained by substituting $t = -x$ in the first integral, then renaming the variable to x and interchanging the limits. Noting that $\sin(-x) = -\sin x$, we have

$$\begin{aligned} f(x) + f(-x) &= \frac{x^2}{1 + \sin x + \sqrt{1 + \sin^2 x}} + \frac{x^2}{1 - \sin x + \sqrt{1 + \sin^2 x}} \\ &= \frac{x^2\left(2 + 2\sqrt{1 + \sin^2 x}\right)}{\left(1 + \sqrt{1 + \sin^2 x}\right)^2 - \sin^2 x} = \frac{x^2\left(2 + 2\sqrt{1 + \sin^2 x}\right)}{2 + 2\sqrt{1 + \sin^2 x}} = x^2. \end{aligned}$$

Hence, the integral to be calculated is equal to

$$\int_0^{\pi} x^2\, dx = \frac{\pi^3}{3}.$$

6) (SENIOR 6) For each $n \geq 0$ let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!} \quad \text{and} \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n b_n$ is an integer.

Source: Problem 3, Second Day, International Mathematics Competition for University Students. See
`http://www.imc-math.org.uk/index.php?year=2002&item=info`

Solution: Write

$$f_n(x) = \sum_{k=0}^{\infty} \frac{k^n x^k}{k!}.$$

Then $a_n = f_n(1)$ and $b_n = f_n(-1)$. It is clear from the Taylor series of e^x that $f_0(x) = e^x$. Furthermore, it is easy to see that

$$x \frac{d}{dx} f_n(x) = f_{n+1}(x).$$

Hence it follows that

$$f_n(x) = P_n(x)e^x,$$

where $P_n(x)$ is polynomial such that $P_0(x) = 1$ and

$$P_{n+1}(x) = x\left(P_n(x) + \frac{d}{dx}P_n(x)\right).$$

It is clear that the coefficients of $P_n(x)$ are integers, and so

$$a_n b_n = f_n(1)f_n(-1) = P_n(1)e^1\, P_n(-1)e^{-1} = P_n(1)P_n(-1)$$

is an integer for any $n \geq 0$.

7) (SENIOR 7) Given real numbers a_k for $k \geq 1$, assume that

$$\sum_{k=1}^{\infty} a_k$$

converges. Prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{k}$$

converges.

Source: Problem 15, p. 317, David V. Widder, *Advanced Calculus*, Second Edition, Dover, New York, 1989.

Solution: The following more general result is true:

Let a_k and b_k for $k \geq 1$ be complex numbers such that

$$\lim_{k\to\infty} b_k = 0 \tag{1}$$

and

$$\sum_{k=1}^{\infty} |b_k - b_{k+1}| < \infty. \tag{2}$$

Assume that there is a real number B such that

$$\left|\sum_{k=1}^{N} a_k\right| < B \tag{3}$$

for all $N \geq 1$. Then the series

$$\sum_{k=1}^{\infty} a_k b_k \tag{4}$$

converges.

This result is the Generalized Dirichlet Convergence Test. In the original version of the Dirichlet Test, instead of (2) one assumes that b_k is real and $b_k \geq b_{k+1} > 0$ for all $k \geq 1$. The Alternating Series Test and the statement of the problem are both consequences of the original version of the Dirichlet Test. Indeed, one obtains the Alternating Series Test if one takes $a_k = (-1)^{k+1}$, and one obtains the result stated in the problem if one takes the $b_k = 1/k$. We will comment on the role of the Generalized Dirichlet Test in number theory below.

To show the above result, write

$$c_n = \sum_{k=1}^{n} a_k \qquad (n \geq 0).$$

Then $a_n = c_n - c_{n-1}$, so, given integers M and N with $0 \leq M < N$ we have

$$\begin{aligned}\sum_{n=M+1}^{N} a_n b_n &= \sum_{n=M+1}^{N} (c_n - c_{n-1}) b_n \\ &= c_N b_{N+1} - c_M b_{M+1} + \sum_{n=M+1}^{N} c_n (b_n - b_{n+1});\end{aligned}$$

the last equation can be easily checked by noting that each term in the middle member is matched by exactly one member on the right-hand side. An equation of this type is called partial summation, or Abel rearrangement, named after the Norwegian mathematician Niels Henrik Abel. Therefore

$$\begin{aligned}\Big|\sum_{n=M+1}^{N} a_n b_n\Big| &\leq |c_N b_{N+1}| + |c_M b_{M+1}| + \sum_{n=M+1}^{N} |c_n||b_n - b_{n+1}| \\ &\leq B\Big(|b_{N+1}| + |b_{M+1}| + \sum_{n=M+1}^{N} |b_n - b_{n+1}|\Big) \qquad (0 \leq M < N);\end{aligned}$$

the second inequality follows in view of (3). Making $M \to \infty$, the limit of the right-hand side is 0 in view of (1) and (2). This shows that the series in (4) indeed converges.

Note. A Dirichlet series is a sum

$$\sum_{n=1}^{\infty} a_n n^{-s}, \tag{5}$$

where the coefficients a_n for $n \geq 1$ are given complex numbers. Johann Peter Gustav Lejeune Dirichlet used these eponymous[2] series to establish his famous result that if an arithmetic progression with integer terms contains two relatively prime integers then it contains infinitely many prime numbers. Dirichlet considered these series only for real s; somewhat later, Georg Friedrich Bernhard Riemann used them with complex s in his study of prime numbers. The basic convergence result for Dirichlet series is the following:

[2]I.e., series named after him (later, by others), that is, Dirichlet series.

If (5) *converges for* $s = s_0$ *with some complex* s_0*, then it also converges for all complex* s *with* $\Re s > \Re s_0$*.*

This is a direct convergence of the Generalized Dirichlet Test. Indeed, assume that

$$\sum_{n=1}^{\infty} a_n n^{-s_0}$$

converges. Then

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n n^{-s_0}\, n^{-(s-s_0)}.$$

Assuming $\Re(s - s_0) > 0$, we have

$$\begin{aligned} |n^{-(s-s_0)} - (n+1)^{-(s-s_0)}| &= \left| \int_n^{n+1} (s - s_0) t^{-(s-s_0)-1}\, dt \right| \\ &\leq \left|(s - s_0) n^{-(s-s_0)-1}\right| = |s - s_0| n^{-\Re(s-s_0)-1}. \end{aligned}$$

Since the series

$$\sum_{n=1}^{\infty} n^{-\Re(s-s_0)-1}$$

is convergent (e.g., by the Integral Test), the Generalized Dirichlet Test implies that the series in (5) is also convergent. If we assume that s and s_0 are real, the same conclusion follows also from the original Dirichlet Test.