

SENIOR PRIZE EXAM  
SPRING 2017

1) Show that for any integer  $n$ , the number  $n^3 + 11n$  is divisible by 6.

2) In a class of 34 students, there are 17 females and 17 males. Assume they are all sitting at a round table. Show that there is at least one (male or female) student with two female neighbors.

3) Let  $p$  and  $q$  be positive integers, and assume that all solutions of the equations

$$x^2 + px - q = 0 \quad \text{and} \quad x^2 + px + q = 0$$

are integers. Show that there are nonzero integers  $a$  and  $b$  such that  $p^2 = a^2 + b^2$ .

4) Show that

$$\sin \frac{\pi}{10} \sin \frac{3\pi}{10} = \frac{1}{4}.$$

5) Calculate the integral

$$\int_{-\pi}^{\pi} \frac{x^2}{1 + \sin x + \sqrt{1 + \sin^2 x}} dx.$$

6) For each  $n \geq 0$  let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!} \quad \text{and} \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that  $a_n b_n$  is an integer.

7) Given real numbers  $a_k$  for  $k \geq 1$ , assume that

$$\sum_{k=1}^{\infty} a_k$$

converges. Prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{k}$$

converges.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE  
<http://www.sci.brooklyn.cuny.edu/~mate/prize/2017/>