All Problems on the Prize Exams
Spring 2018

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Show that the product of four consecutive integers plus 1 is always a square of an integer.

Source: Problem 185, p. 80, Középiskolai Matematikai Lapok, Vol. 3, No. 5, January 1896.

`http://db.komal.hu/scan/1896/01/89601080.g4.png`

Solution: The main challenge in the problem is to obtain the result without messy calculations. Let the four consecutive integers be $k-1$, k, $k+1$, and $k+2$. Then we have

$$\begin{aligned}(k-1)k(k+1)(k+2) &= (k-1)(k+2)\,(k^2+k) = (k^2+k-2)(k^2+k)\\ &= \big((k^2+k-1)-1\big)\big((k^2+k-1)+1\big) = (k^2+k-1)^2-1,\end{aligned}$$

showing that indeed

$$(k-1)k(k+1)(k+2)+1 = (k^2+k-1)^2$$

is the square of an integer.

2) (JUNIOR 2 and SENIOR 2) Let a, b be real numbers and assume $b \ne 0$. Determine p such that

$$x^2 + 2(2b-p)x + p^2 + 4ab$$

is the square of a polynomial of x.

Source: Problem 440, p. 67, Középiskolai Matematikai Lapok, Vol. 5, No. 4, December 1897.

`http://db.komal.hu/scan/1897/12/89712067.g4.png`

Solution: Assuming $A \ne 0$, for a quadratic polynomial

$$Ax^2 + Bx + C$$

to be a complete square, the equation

$$Ax^2 + Bx + C = 0$$

must have exactly one solution, i.e., its discriminant $B^2 - 4AC$ must be 0. In the present case we have $A = 1$, $B = 2(2b-p)$, and $C = p^2 + 4ab$, so to ensure that the expression given in the problem is a complete square, we must have

$$0 = B^2 - 4AC = 4\big((2b-p)^2 - 4(p^2+4ab),$$

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

i.e., after dividing by 4 and evaluating the square, we obtain

$$(4b^2 - 4bp + p^2) - (p^2 + 4ab) = 0,$$

that is,

$$b^2 - bp - ab = 0.$$

Since we assumed that $b \neq 0$, this is equivalent to

$$p = b - a.$$

3) (JUNIOR 3 and SENIOR 3) Define a sequence as follows: $a_1 = 2$, and $a_{n+1} = a_n^2 - a_n + 1$ for all $n \geq 1$. Show that if $m > n \geq 1$ then a_m and a_n are relatively prime (i.e., their greatest common divisor is 1).

Source: Problem 6 of the 2010 Eastern Oregon University Mathematics Competition, See

`http://www.problemcorner.or://www.eou.edu/math/competition/`

Solution: Let $k > 1$, $n \geq 1$ and assume that $k \mid a_n$ (in words: k is a divisor of a_n) or that $k \mid a_n - 1$. We claim that then $k \mid a_m - 1$ for all $m > n$.

Indeed, this is true for $m = n + 1$, since $a_{n+1} - 1 = a_n(a_n - 1)$. Assume that for some $m > n$ we have $k \mid a_m - 1$. Then $k \mid a_m(a_m - 1) = a_{m+1} - 1$. Hence, our claim follows for all $m > n$ by induction.

Thus if $k > 1$ and $1 < n < m$ and $k \mid a_n$, then $k \nmid a_m$ since $k \mid a_m - 1$.

4) (JUNIOR 4) Let a and b be integers neither of which is divisible by 3. Show that $a^6 - b^6$ is divisible by 9.

Source: Problem 1163, p. 13, Középiskolai Matematikai Lapok, Vol. 10, No. 9, April 1903.

`http://db.komal.hu/scan/1903/04/90304213.g4.png`

Solution: If a is not divisible by 6 then $a^6 - 1$ is divisible by 9. There are many simple ways of seeing this, but before we embark on one of these ways, we would like to point out that this is a special case of a theorem of Euler:

Euler's theorem. *For a positive integer n, let $\phi(n)$ be the number of positive integers $< n$. Given any integer m relatively prime to n, the number n is a divisor of $m^{\phi(n)} - 1$.*

Given that $\phi(9) = 6$, it follows that $a^6 - 1$ is divisible by 9 unless 3 is a divisor of a. Euler's theorem generalizes an earlier theorem of Fermat, which makes the same conclusion in case n is a prime. Now if neither a nor b is divisible by 3, we can see that $a^6 - b^6 = (a^6 - 1) - (b^6 - 1)$ is divisible by 9.

To conclude, assume a is not divisible by 3. We will show that $a^6 - 1$ is divisible by 9 without appealing to Euler's theorem. We have $a = 3k \pm 1$ for some integer k, and so

$$a^2 = (3k \pm 1) = 9k^2 \pm 6k + 1 = 3l + 1$$

with $l = 3k^2 \pm 2k$. Hence

$$a^6 = (a^2)^3 = (3l + 1)^3 = (3l)^3 + 3(3l)^2 + 3(3l) + 1 = 9(3l^3 + 3l^2 + l) + 1.$$

Thus, $a^6 - 1$ is indeed divisible by 9.

5) (JUNIOR 5) Determine all primes p such that $p^2 + 2$ is also a prime.

Source: Problem 127, p. 60, Középiskolai Matematikai Lapok, Vol. 1/4, December 1, 1947, proposed by Tibor Szele:

`http://db.komal.hu/scan/1947/12/94712060.g4.png`

Solution: We must have $p = 3$. Indeed, if $p \neq 3$, then $p^2 - 1$ is divisible by 3 according to Fermat's theorem. Therefore, $p^2 + 2 = (p^2 - 1) + 3$ is also divisible by 3; hence $p^2 + 2$ is not a prime. If $p = 3$ then $p^2 + 2 = 11$ is also a prime, so $p = 3$ is the only prime that satisfies the requirements.

6) (JUNIOR 6) Show that we have

$$2x < \sin x + \tan x$$

for every x with $0 < x < \pi/2$.

Source: Problem 4, proposed by Zoltán Bogdán, Cegléd, Hungary, XIII. Nemzetközi Magyar Matematika Verseny (13th International Hungarian Mathematics Competition), Nagydobrony (Velyka Dobron, Ukraine), March 15–20, 2004. See

`http://nmmv.berzsenyi.hu/`

Solution by trigonometry: We have $\alpha < \tan\alpha$ for any α with $0 < \alpha < \pi/2$. Therefore $x/2 < \tan(x/2)$ for x in the given range. Therefore, it is sufficient to show that

$$4\tan\frac{x}{2} < \sin x + \tan x \qquad \left(0 < x < \frac{\pi}{2}\right).$$

Writing $t = \tan(x/2)$, we have $0 < t < 1$ for x with $0 < x < \pi/2$ and

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \tan x = \frac{2t}{1-t^2};$$

hence it is sufficient to prove that

$$4t < \frac{2t}{1+t^2} + \frac{2t}{1-t^2} \qquad (0 < t < 1).$$

Dividing both sides by $2t$ and multiplying them by $(1 + t^2)(1 - t^2)$, we obtain the equivalent inequality

$$2(1 - t^4) < 2 \qquad (0 < t < 1).$$

This inequality is obviously true, establishing the assertion.

Solution by elementary geometry: Let O be the center of the coordinate system, and let A be the point with coordinates $(1, 0)$, let θ be an angle with $0 < \theta < \pi/2$, let B be the point in the first quadrant on the unit circle such that $\angle AOB = \theta$; let C be the intersection of the line OB and the line $x = 1$. The area of sector $\widehat{ABO}$ of the unit circle is $\theta/2$. The area of the triangle $\triangle ABO$ is $(1/2)\sin\theta$. Finally, the area of the triangle $\triangle ACO$ is $(1/2)\tan\theta$ Hence, the inequality

$$2\theta < \sin\theta + \tan\theta$$

can be reformulated as saying that the area of the circular segment bounded by the line segment $\overline{AB}$ and the arc $\widehat{AB}$ of the unit circle is less than the area of the part of the triangle ACO outside the circle sector $\widehat{ABO}$.

To show this, let D be the point on the line $x = 1$ in the first quadrant such that $\angle AOD = \theta/2$. Let E be the point on the line $x = 1$ such that $OD \parallel BE$, and, finally, let F be the intersection of the chord AB and the line OD. It is easy to see that $\overline{BE} = 2\overline{FD}$; hence, the area of the triangle $\triangle DBE$ equals the area of the triangle $\triangle DBA$. The line BD is tangent to the unit circle since the triangles $\triangle BDO$ and $\triangle ADO$ are congruent. Hence the triangle $\triangle DBE$ lies entirely outside circular segment bounded by the line segment $\overline{AB}$ and the arc $\widehat{AB}$ of the unit circle; since the triangle $\triangle DBA$ includes this circular segment, the assertion follows.

Solution by calculus: The inequality to be established can be written as

$$\int_0^x 2\,dt < \int_0^x (\cos t + \sec^2 t)\,dt \qquad \left(0 < x < \frac{\pi}{2}\right),$$

and to prove this it is enough to show that

$$2 < \cos t + \sec^2 t \qquad \left(0 < t < \frac{\pi}{2}\right).$$

Writing $y = \cos t$, this inequality is equivalent to

$$2 < y + \frac{1}{y^2} \qquad (0 < y < 1),$$

or else

$$0 < y^3 - 2y^2 + 1 \qquad (0 < y < 1).$$

To see this latter, notice that the right-hand side can be factored as $(y-1)(y^2-y-1)$, and both factors are negative when $0 < y < 1$. For the second factor, this can be seen by observing that its value is negative when $y = 0$ and when $y = 1$, and the coefficient of y^2 is positive; therefore it is negative for all y with $0 < y < 1$.

7) (JUNIOR 7) Let $P(x)$ be a polynomial of positive degree with integer coefficients. Let n_1 be the number of distinct integer roots of $P(x) = 1$, and n_2 the number of distinct integer roots of $P(x) = -1$. Prove that if both n_1 and n_2 are positive, then $n_1 + n_2 \leq 5$.

Source: Problem 5 of the 2017 Rutgers Undergraduate Problem Solving Competition. See `http://www.problemcorner.or://sites.math.rutgers.edu/~prize/`

Solution: The key to the solution of this problem is a special case of the Rational Roots Theorem: Let $F(x)$ be a nonconstant polynomial with integer coefficients, and let m be an integer such that $F(m) = 0$. Then m is a divisor of the constant term of $F(x)$.[1]

In a more symmetric formulation of the problem, let $Q(x)$ and $R(x)$ be polynomials with integer coefficients such that $Q(x) - R(x) = \pm 2$. Let n_Q be the number of distinct integer roots of $Q(x) = 0$, and n_R the number of distinct integer roots of $R(x) = 0$, and assume that n_Q and n_R are positive. Then $n_Q + n_R \leq 5$. In this formulation, we can take $Q(x) = P(x) - 1$ and $R(x) = P(x) + 1$, else $Q(x) = P(x) + 1$ and $R(x) = P(x) - 1$.

Assume α is an integer that is a root of $Q(x) = 0$; then 0 is a root of $Q(x + \alpha) = 0$. Hence the constant term of $Q(x + \alpha)$ is 0. Therefore, the constant term of $R(x + \alpha)$ is ± 2. Now, if β is an integer root of $R(x) = 0$, then $\beta - \alpha$ is an integer root of $R(x + \alpha) = 0$; So $\beta - \alpha$ is the divisor of

[1] In order not to hedge this statement with various exceptions involving 0, one needs to accept the view that any integer is a divisor of 0; in particular, 0 itself is a divisor of 0 (but not of any other integer).

the constant term of this latter polynomial, i.e., it is a divisor of ± 2. Hence $\beta - \alpha$ is one of the numbers ± 1 and ± 2. Therefore, $1 \le |\beta - \alpha| \le 2$.

Now assume that γ is the smallest integer among the roots of the equations $Q(x) = 0$ and $R(x) = 0$; without loss of generality, we may assume that γ is a root of $Q(x) = 0$. Then the only integer roots of the equation $R(x) = 0$ can be $\gamma + 1$ or $\gamma + 2$; since this equation has an integer root, at least one of these numbers must be a root. So any root of the equation of $Q(x) = 0$ is at most this root plus 2; thus any root of the latter equation is $\le (\gamma + 2) + 2 = \gamma + 4$. Thus, all the integer roots of the equations $Q(x) = 0$ and $R(x) = 0$ are among the numbers γ, $\gamma + 1$, $\gamma + 2$, $\gamma + 3$, and $\gamma + 4$. Since these equations cannot have common roots, we can conclude that indeed $n_Q + n_R \le 5$.

8) (SENIOR 4) Find the largest power of 3 that divides 100!.

Solution: If p is a prime and n is a positive integer, the largest power of p that divides $n!$ has exponent

$$\sum_{k=1}^{n} \left\lfloor \frac{n}{p^k} \right\rfloor;$$

of course, the terms in this sum for which $p^k > n$ are 0. The reason for this is that the number of positive integers $m \le n$ that are divisible by p^k is $\lfloor n/p^k \rfloor$. If the largest power of p that divides m is p^r, then m is counted exactly r times in the above sum, namely for each k with $1 \le k \le r$. The above formula is due to Adrien-Marie Legendre.

In the specific case, $\lfloor 100/3 \rfloor = 33$, $\lfloor 100/3^2 \rfloor = \lfloor 33/3 \rfloor = 11$, $\lfloor 100/3^3 \rfloor = \lfloor 11/3 \rfloor = 3$, $\lfloor 100/3^4 \rfloor = \lfloor 3/3 \rfloor = 1$, $\lfloor 100/3^5 \rfloor = \lfloor 1/3 \rfloor = 0$. Thus, for $n = 100$, the above sum is $33 + 11 + 3 + 1 = 48$. Thus, $3^{48} \mid 100!$ but $3^{49} \nmid 100!$.

9) (SENIOR 5) Given positive real numbers a, b, and c such that $abc = 1$, show that

$$\frac{a+b+c+3}{4} \ge \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

Source: 2015 European Mathematical Cup, Senior Division, Problem 2. Proposed by Dimitar Trenevski. See

`https://artofproblemsolving.com/community/c388198_2015 _european_mathematica_cup`

Solution: We will show that under the assumptions we have

$$\frac{a+1}{4} \ge \frac{1}{b+c}. \tag{1}$$

By taking cyclic permutations of $\langle a, b, c \rangle$, two other similar inequalities can be obtained:

$$\frac{b+1}{4} \ge \frac{1}{c+a}$$

and

$$\frac{c+1}{4} \ge \frac{1}{a+b}.$$

Adding these three inequalities, the result follows.

We proceed to establish (1). Noting that $abc = 1$, this inequality can be written equivalently as

$$\frac{1}{4bc} - \frac{1}{b+c} + \frac{1}{4} \ge 0.$$

Multiplying both sides by $4bc(b+c)$, this is equivalent to

$$b + c - 4bc + (b+c)bc \geq 0,$$

or else as

$$b(1 - 2c + c^2) + c(1 - 2b + b^2) \geq 0.$$

Since the expressions in parentheses are complete squares, the last inequality follows. As (1) is equivalent to this inequality, it also follows.

10) (SENIOR 6) Show that the limit

$$\lim_{n\to\infty} \left(-2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right)$$

exists.

Source: Problem 6, University of Connecticut Undergraduate Calculus Competition, 2014. See `https://undergradactivities.math.uconn.edu/calculus-competition/`

Solution: We have

$$\int_0^n \frac{1}{\sqrt{x}}\,dx = \lim_{\epsilon\searrow 0} \int_\epsilon^n \frac{1}{\sqrt{x}}\,dx = \lim_{\epsilon\searrow 0} 2\sqrt{x}\Big|_{x=\epsilon}^{x=n} = 2\sqrt{n}.$$

Hence

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \int_0^n \frac{1}{\sqrt{x}}\,dx = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} - \int_{k-1}^{k} \frac{1}{\sqrt{x}}\,dx \right). \tag{1}$$

Note that on the left-hand side, we wrote the sum first, as opposed to the way we wrote the same expression in the question. According to conventions of mathematical notation, the scope of the summation sign ends at the first plus or minus sign after the sum sign; in the formulation of the problem we wanted to avoid misunderstandings in case the reader is not familiar with this convention.

It is easy to show that the series on the right-hand side converges. Indeed, we have

$$\frac{1}{\sqrt{k}} - \int_{k-1}^{k} \frac{1}{\sqrt{x}}\,dx = \int_{k-1}^{k} \frac{1}{\sqrt{k}}\,dx - \int_{k-1}^{k} \frac{1}{\sqrt{x}}\,dx = \int_{k-1}^{k} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{x}} \right) dx.$$

In the first integral of the second member, we are integrating a constant; the purpose of writing things this way was to bring everything under the same integral sign. The integrand is easy to estimate, say, by the mean-value theorem of differentiation. According to this, for x with $0 < x < k$ we have

$$\geq \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{x}} = -\frac{1}{2\xi^{3/2}}(k - x)$$

for some ξ with $x < \xi < k$. For $k > 1$ and $k - 1 \leq x < k$ we therefore have

$$0 > \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{x}} > -\frac{1}{2(k-1)^{3/2}}$$

Thus, the for $k > 1$ the kth term of the series on the right-hand side of (1) is between 0 and $-(k-1)^{-3/2}/2$. Since the series

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)^{3/2}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

is convergent by the integral test, it follows that the series on the right-hand side of (1) converges by the comparison test.

Note. The value of the limit in the question is $\zeta(1/2) \approx -1.4603545$, where $\zeta(s)$ is the Riemann zeta function; the problem in the University of Connecticut competition asked about minus the above limit; we changed it so as to be faithful to the important context of the problem. The Riemann zeta function for the complex variable s is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This series is convergent for $\Re s > 1$, and so it only defines the zeta function for $\Re s > 1$. The function it represents can be extended to the whole complex plane by *analytic continuation*; it is differentiable *holomorphic* everywhere except at the point $s = 1$, where it has a pole (it becomes infinite). This analytic continuation defines the Riemann zeta function on the rest of the complex plane. For s with $0 < \Re s < 1$ it can be described by the formula

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k^s} - \int_0^n \frac{1}{n^s} \right).$$

For $s = 1/2$ this limit is identical to the one featured in the problem.

The Riemann zeta function plays an important role in the study of prime numbers. The Riemann Hypothesis, one of the most famous unsolved problems in mathematics, asserts that all zeros in the the strip $0 < \Re s < 1$ of the zeta function lie on the line $\Re s = 1/2$, called the critical line. These zeros are called the nontrivial zeros of the zeta function; in addition to the nontrivial zeros, the zeta function has trivial zeros at all negative even integers ($-2, -4, -6, \dots$). From the location of the nontrivial zeros, one can make important conclusions about the distribution of prime numbers. In fact, for $x > 0$ writing $\pi(x)$ for the number of prime numbers $\le x$, the Prime Number Theorem asserts that

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1$$

($\log x$ denotes the natural logarithm – the notation $\ln x$ is rarely used in mathematics, but it is used in other sciences applying mathematics). The Prime Number Theorem is a consequence of the result that the Riemann zeta function has no zeros on the line $\Re s = 1$.

11) (SENIOR 7) Let f be a twice differentiable real-valued function defined on the real line, and assume that $f(0) = 0$. Assume further that f'' is continuous. Prove that there exists a $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

Source: Problem 2, International Mathematics Competition for University Students, 2013, Blagoevgrad, Bulgaria Day 1, August 8, 2013. See

Solution: The assumption about the continuity of f'' is unnecessary, but showing this involves some subtle points that most students even with a solid background in the undergraduate course Advanced Calculus are likely to miss. In the solution below, we do not assume that f'' is continuous, and we will be careful to point out the subtle issues.[2]

If $f''(0) = 0$, then $\xi = 0$ satisfies the requirements. Therefore we may assume that $f''(0) > 0$; indeed, if $f''(0) < 0$, we may replace $f(x)$ with $-f(x)$. Thus, putting

$$F(x) = f''(x) - f(x)(1 + 2\tan^2 x),$$

we have $F(0) > 0$. Thus, it will be enough to show that there is an $x_0 \in (-\pi/2, \pi/2)$ such that $F(x_0) < 0$. One might be tempted to think that this is a consequence of the Intermediate-Value Theorem for continuous functions, but in fact it is not assumed that $f''(x)$ is continuous. We have, however, the following

Intermediate-Value Theorem for Derivatives. *be Let $a < b$, and let G be a function that is differentiable on the interval $[a, b]$ (also assuming that G is differentiable at a and b), and assume $G'(a) < 0$ and $G'(b) > 0$. Then there is a $\xi \in (a, b)$ such that $G'(\xi) = 0$.*

Proof. We want to emphasize again that G' need not be continuous. As G is differentiable on $[a, b]$, it is continuous there, so it has an absolute minimum on this interval, by the Maximum-Value Theorem.[3] Let x_0 be the place of absolute minimum of G in the interval $[a, b]$. We cannot have $x_0 = a$, since $G'(a) < 0$, so for any $\epsilon > 0$ there is an $x \in (a, a+\epsilon)$ such that $G(a) > G(x)$. Similarly, we cannot have $x_0 = b$, since $G'(b) > 0$. Therefore, $x_0 \in (a, b)$. Hence $G'(x_0) = 0$ by Fermat's Theorem,[4] as we wanted to show. $\square$

An immediate generalization of this is that if G is differentiable in $[a, b]$ and c is such that $G'(a) < c < G'(b)$ or $G'(a) > c > G'(b)$ then there is a $\xi \in (a, b)$ such that $G'(\xi) = c$. To see this, one only needs to apply the theorem just proved to the function $G(x) - cx$ or $cx - G(x)$ replacing G.

Now, F does not look like the derivative of a function, but indeed we have

$$F(x) = \frac{d}{dx}\left(f'(x) - \int_0^x f(t)(1 + 2\tan^2 t)\,dt\right) \qquad \bigl(x \in (-\pi/2, \pi/2)\bigr)$$

by the Fundamental Theorem of Calculus, since the integrand is continuous (because it is differentiable) on $(-\pi/2, \pi/2)$.[5]

In order to show that there is an $x_0 \in (-\pi/2, \pi/2)$, such that $F(x_0) < 0$, assume on the contrary that

$$(1) \qquad F(x) > 0 \qquad \text{for all} \quad x \in (-\pi/2, \pi, 2).$$

[2] The source mentioned above posted the problem without the assumption of the continuity of f''. We wonder how the competitors handled this issue. We have not read the published solution to the problem, in accordance with our policy that we never read the posted solutions to problems to make sure that our solution is independent of whatever was posted.

[3] The Maximum-Value Theorem asserts that a function continuous on a closed interval has a maximum in that interval. If $-f$ has a maximum, then f has a minimum, so the Maximum-Value Theorem is also used to justify the existence of a minimum.

[4] Fermat's Theorem says that if f has a local extremum at x_0 and $f'(x_0)$ exists, then $f'(x_0) = 0$.

[5] Observe that a continuous function is always the derivative of its integral. Therefore, the Intermediate-Value Theorem for Derivatives can be generalized to say that the sum of the derivative of a function and of a continuous function satisfies the Intermediate-Value Property. In fact, the Intermediate-Value Theorem for continuous functions is itself a consequence of the Intermediate-Value Theorem for Derivatives.

We claim that this assumption implies that $f(\pi/2) \le 0$ and $f(-\pi/2) \le 0$. Assume, on the contrary, for example, that $f(-\pi/2) > 0$. As f is continuous, and so $\lim_{x\to\pi/2} f(x) = f(\pi/2)$, this implies that

$$\lim_{x\nearrow\pi/2} f(x)(1+\tan^2 x) = +\infty.$$

Then (1) implies that we also have

$$\lim_{x\nearrow\pi/2} f''(x) = +\infty;$$

this, however, is not the case. Indeed, by the Intermediate-Value Theorem for Derivatives, there is a sequence of reals $x_n < \pi/2$ such that $x_n \to \pi/2$ and $f''(x_n) < f''(\pi/2) + 1$.[6] This contradiction shows that $f(\pi/2) \le 0$. A similar argument shows that also $f(-\pi/2) < 0$.

The assumptions that $f(0) = 0$ and $f''(0) > 0$ implies that for every $\epsilon > 0$ the interval $(-\epsilon, \epsilon)$ contains a point x for which $f(x) > 0$; otherwise, $x = 0$ would be a place of local maximum of f, and so we would have $f''(0) \le 0$ by the second derivative test for extrema.[7] Let x_0 be a point where f assumes an absolute maximum on the interval $[-\pi/2, \pi/2]$. Then $f(x_0) > 0$, and so $x_0 \in (-\pi/2, \pi/2)$ (because $f(\pi/2) \le 0$ and $f(-\pi/2) \le 0$). We also have $f''(x_0) \le 0$ by the second derivative test, and so $F(x_0) < 0$.[8] This contradicts (1), verifying the assertion of the problem.

[6]This is the subtle point we expect most students to miss even if they are aware of the Intermediate-Value Theorem for Derivatives. In fact, not only we need not have

$$\lim_{x\nearrow\pi/2} f''(x) = f(\pi/2),$$

$f''(x)$ need not even be bounded near $\pi/2$.

[7]Indeed, if $f(x) = 0$ and $f(x) \le 0$ for $x \in (-\epsilon, \epsilon)$, then 0 is a place is a place of local maximum of f, and so $f'(0) = 0$ by Fermat's Theorem. Therefore, the second derivative test says that $x = 0$ is a place of is a strict local minimum, since $f''(0) > 0$. That is, there is an $\epsilon' > 0$ such that $f(x) > 0$ for all $x \ne 0$ in the interval $(-\epsilon', \epsilon')$.

[8]It was important to make sure that the place of maximum x_0 is inside the interval $[-\pi/2, \pi/2]$, since the second derivative test is not applicable for extrema assumed at the endpoint of a closed interval.