Junior Prize Exam
Spring 2018

1) Show that the product of four consecutive integers plus 1 is always a square of an integer.

2) Let a, b be real numbers and assume $b \neq 0$. Determine p such that

$$x^2 + 2(2b - p)x + p^2 + 4ab$$

is the square of a polynomial of x.

3) Define a sequence as follows: $a_1 = 2$, and $a_{n+1} = a_n^2 - a_n + 1$ for all $n \geq 1$. Show that if $m > n \geq 1$ then a_m and a_n are relatively prime (i.e., their greatest common divisor is 1).

4) Let a and b be integers neither of which is divisible by 3. Show that $a^6 - b^6$ is divisible by 9.

5) Determine all primes p such that $p^2 + 2$ is also a prime.

6) Show that we have

$$2x < \sin x + \tan x$$

for every x with $0 < x < \pi/2$.

7) Let $P(x)$ be a polynomial of positive degree with integer coefficients. Let n_1 be the number of distinct integer roots of $P(x) = 1$, and n_2 the number of distinct integer roots of $P(x) = -1$. Prove that if both n_1 and n_2 are positive, then $n_1 + n_2 \leq 5$.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2018/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{AMS}$-TeX was used for typesetting.