Senior Prize Exam
Spring 2018

1) Show that the product of four consecutive integers plus 1 is always a square of an integer.

2) Let a, b be real numbers and assume $b \neq 0$. Determine p such that

$$x^2 + 2(2b - p)x + p^2 + 4ab$$

is the square of a polynomial of x.

3) Define a sequence as follows: $a_1 = 2$, and $a_{n+1} = a_n^2 - a_n + 1$ for all $n \geq 1$. Show that if $m > n \geq 1$ then a_m and a_n are relatively prime (i.e., their greatest common divisor is 1).

4) Find the largest power of 3 that divides 100!.

5) Given positive real numbers a, b, and c such that $abc = 1$, show that

$$\frac{a+b+c+3}{4} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

6) Show that the limit

$$\lim_{n\to\infty}\left(-2\sqrt{n} + \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\right)$$

exists.

7) Let f be a twice differentiable real-valued function defined on the real line, and assume that $f(0) = 0$. Assume further that f'' is continuous. Prove that there exists a $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2\xi).$$

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2018/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.