All Problems on the Prize Exams
Spring 2019
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Find all pairs of positive integers (n, k) such that

$$n! + 8 = 2^k.$$

Source: Problem 1, Australian Mathematical Olympiad 2018. See
`https://www.amt.edu.au/wp-content/uploads/2018_AMO.pdf`
Solution: According to the given equation, we have

$$n! = 2^k - 8 = 8(2^{k-3} - 1).$$

This means that $n!$ is divisible by 8, but by no higher power of 2. The only values of n that satisfy this requirement are 4 and 5. Given that $4! = 24$ and $5! = 120$, we can see that for $n = 4$ and $n = 5$ the quantity $n! + 8$ is indeed a power of 2. Hence the only pairs of values that satisfy the required equation are $(n, k) = (4, 5)$ and $(5, 7)$.

2) (JUNIOR 2 and SENIOR 2) Let x and y be integers. Show that

$$x(x + 1) \neq 2(5y + 2).$$

Source: Problem 570, p. 155, Középiskolai Matematikai Lapok, Vol. XVIII, No. 5, May 1959, submitted by István Möller.
`http://db.komal.hu/scan/1959/05/95905155.g4.png`
Solution: The right-hand side is $2(5y + 2) = 10y + 4 \equiv 4 \mod 5$. It is easy to see that the left-hand side does not satisfy this congruence. Indeed, if $x \equiv k \mod 5$, then $x^2 + x \equiv k^2 + k \mod 5$, and for $k = 0, 1, 2, 3, 4$ we have $k^2 + k = 0, 2, 6, 10, 17$ in turn, and none of these is congruent to 4 modulo 5.

3) (JUNIOR 3 and SENIOR 3) Given positive real numbers a, b, and c, prove that

$$(a + b)(b + c)(c + a) \geq 8abc.$$

Source: Problem 392, p. 30, Középiskolai Matematikai Lapok, New Series, Vol. XIV, No. 1, January 1957, submitted by Gyula Veszely.
`http://db.komal.hu/scan/1957/01/95701030.g4.png`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

Solution: For any positive x and y we have $(x+y)/2 \geq \sqrt{xy}$ according to the inequality between the arithmetic and geometric means. Hence

$$\frac{a+b}{2} \cdot \frac{b+c}{2} \cdot \frac{c+a}{2} \geq \sqrt{ab} \cdot \sqrt{bc} \cdot \sqrt{ca} = abc,$$

implying the above inequality.

4) (JUNIOR 4) Given an isosceles trapezoid, consider a triangle with sides equal to the leg, the diagonal, and the geometric mean of the parallel sides of the trapezoid. Show that this triangle is a right triangle.

In other words, given a trapezoid $ABCD$ with $AB \parallel CD$, $AD = BC$, and $AD \nparallel BC$ unless $\angle DAB$ is a right angle, show that a triangle with sides of lengths AD, BD, and $\sqrt{AB \cdot CD}$ is a right triangle.

Source: Problem 395, p. 30, Középiskolai Matematikai Lapok, New Series, Vol. XIV, No. 1, January 1957, submitted by Erik Dux.

`http://db.komal.hu/scan/1957/01/95701030.g4.png`

Solution: We may assume that the trapezoid is placed in a Cartesian coordinate system with the vertices $A(a,0)$, $B(-a,0)$, $C(-b,h)$, and $D(b,h)$, where a, b, and h are positive numbers. We have

$$AD^2 = h^2 + (a-b)^2$$

and

$$BD^2 = h^2 + (a+b)^2.$$

Hence

$$BD^2 - AD^2 = 4ab = AB \cdot CD.$$

Hence, the triangle with the described sides is a right triangle, according to the (converse of the) Pythagorean theorem.

5) (JUNIOR 5) The function f defined everywhere on the real line satisfies

$$f(x+1) = \frac{1+f(x)}{1-f(x)}$$

for all real x. Show that f is periodic; that is, there is a positive real number T such that $f(x+T) = f(x)$ for all real x.

Source: Problem 1997, p. 30, Középiskolai Matematikai Lapok, Vol. 51, No. 1, September 1975,

`http://db.komal.hu/scan/1975/09/97509030.g4.png`

Solution: For any real x we have

$$f(x+2) = \frac{1+f(x+1)}{1-f(x+1)} = \frac{1+\frac{1+f(x)}{1-f(x)}}{1-\frac{1+f(x)}{1-f(x)}} = \frac{\big(1-f(x)\big)+\big(1+f(x)\big)}{\big(1-f(x)\big)-\big(1+f(x)\big)} = \frac{2}{-2f(x)} = -\frac{1}{f(x)}.$$

Hence

$$f(x+4) = -\frac{1}{f(x+2)} = -\frac{1}{-\frac{1}{f(x)}} = f(x),$$

establishing the assertion with $T = 4$.

6) (JUNIOR 6) Let x be a nonzero real number such that

$$n = x + \frac{1}{x}$$

is an integer. Show that

$$A = x^4 + x^3 + x^2 + x^{-2} + x^{-3} + x^{-4}$$

is also an integer.

Source: Based on Problem 3 Dániel Arany Mathematical Competition (Hungary) academic year 2011/12, 3rd (final) round, beginners, category II. See

`https://matek.fazekas.hu/index.php?option=com_content&view=article&id=57:arany-daniel-matematikaverseny&catid=26&Itemid=185`

First Solution: To obtain an easier overview of the situation, we will consider the polynomial

$$Ax^4 = x^8 + x^7 + x^6 + x^2 + x + 1.$$

This polynomial can be factored as

$$Ax^4 = (x^2 + x + 1)(x^6 + 1) = (x^2 + x + 1)(x^2 + 1)(x^4 - x^2 + 1);$$

here, the second equation uses the identity

$$t^3 + 1 = (t + 1)(t^2 - t + 1)$$

with $t = x^2$. Thus, we have

$$\begin{aligned} A &= x^{-1}(x^2 + x + 1) \cdot x^{-1}(x^2 + 1) \cdot x^{-2}(x^4 - x^2 + 1) \\ &= \left(x + \frac{1}{x} + 1\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2} - 1\right). \end{aligned}$$

All three factors on the right are integers. Indeed, the first factor is $n + 1$, and the second factor is n. As for the third factor, we have

$$x^2 + \frac{1}{x^2} - 1 = \left(x^2 + 2 + \frac{1}{x^2}\right) - 3 = \left(x + \frac{1}{x}\right)^2 - 3 = n^2 - 3.$$

Second Solution: We have

$$A = (x^4 + x^{-4}) + (x^3 + x^{-3}) + (x^2 + x^{-2}).$$

In order to show that this is an integer, it is sufficient to show that $x^n + x^{-n}$ is an integer for every integer $n \geq 0$. We will show this by induction on n. For $n = 0$ this is obviously true, for $n = 1$ this is true by the assumption. Let $n \geq 2$, and assume that $x^k + x^{-k}$ is an integer for every integer k with $0 \leq k < n$. The general case will be easier to follow if we first discuss a few initial cases.

When $n = 2$, we have

$$(x + x^{-1})^2 = x^2 + 2 + x^{-2} = (x^2 + x^{-2}) + 2.$$

Since the left-hand side is an integer by assumption, $x^2 + x^{-2}$ must also be an integer; thus, the assertion follows for $n = 2$.

When $n = 3$, we have

$$(x + x^{-1})^3 = x^3 + 3x + 3x^{-1} + x^{-3} = (x^3 + x^{-3}) + (x + x^{-1}).$$

Since $x + x^{-1}$ (occurring on both sides of the equation) is an integer by assumption, $x^3 + x^{-3}$ must also be an integer; thus, the assertion follows for $n = 3$.

When $n = 4$, we have

$$(x + x^{-1})^4 = x^4 + 4x^2 + 6 + 4x^{-2} + x^{-4} = (x^4 + x^{-4}) + 4(x^2 + x^{-2}) + 6.$$

Since the expression on the left-hand side and $x^2 + x^{-2}$ are integers (the latter by the induction hypothesis), $x^4 + x^{-4}$ must also be an integer; thus, the assertion follows for $n = 4$.

This much is in fact enough to establish the assertion that A is an integer, solving the problem. Nevertheless, we are going to show the more general assertion that $x^n + x^{-n}$ is an integer under the assumption that $x + x^{-1}$ is an integer. To this end, let $n \geq 2$, and assume that $x^k + x^{-k}$ is an integer for every integer k with $0 \leq k < n$. By the binomial theorem, we have

$$\begin{aligned}
(x + x^{-1})^n &= \sum_{i=0}^{n} \binom{n}{i} x^i x^{-(n-i)} = \sum_{i=0}^{n} \binom{n}{i} x^{2i-n} \\
&= \frac{1}{2}\left(\sum_{i=0}^{n} \binom{n}{i} x^{2i-n} + \sum_{i=0}^{n} \binom{n}{n-i} x^{2(n-i)-n} \right) \\
&= \frac{1}{2}\left(\sum_{i=0}^{n} \binom{n}{i} x^{2i-n} + \sum_{i=0}^{n} \binom{n}{n-i} x^{-(2i-n)} \right) = \frac{1}{2} \sum_{i=0}^{n} \binom{n}{i} \left(x^{2i-n} + x^{-(2i-n)} \right);
\end{aligned} \tag{1}$$

the third equation here is obtained by writing the sum before the equation twice, once in its original form, and the second time with i replaced with $n - i$ in the sum (which gives the same sum); the fifth equation follows since $\binom{n}{i} = \binom{n}{n-i}$.

In the sum on the right-hand side, the terms for i and $n - i$ are the same, so unless $i = n/2$, these terms occur twice. Separating out the terms for $i = 0$ and $i = n$, and for $i = n/2$ if n is even, we find that the right-hand side equals

$$(x^n + x^{-n}) + \sum_{i:1 \leq i < n/2} \binom{n}{i} \left(x^{2i-n} + x^{-(2i-n)} \right)$$

if n is odd, and

$$(x^n + x^{-n}) + \binom{n}{n/2} + \sum_{i:1 \leq i \leq n/2} \binom{n}{i} \left(x^{2i-n} + x^{-(2i-n)} \right)$$

if n is even. The binomial before the sum represents the term for $i = n/2$, when $x^{2i-n} + x^{-(2i-n)} = x^0 + x^0 = 2$ – this cancels against the factor $1/2$ on the right-hand side of (1). Noting that all the terms except for $(x^n + x^{-n})$ in these sums are integers, and the left-hand side of (1) is also an integer, it follows that $x^n + x^{-n}$ is an integer.

7) (JUNIOR 7) Let $n \geq 3$ be an integer. One wants to place n positive integers around a circle in such a way that if a and b are adjacent then either a is a divisor of b or b is a divisor of a, and if a and b are not adjacent then neither is the divisor of the other. Show that this is possible if and only if n is even.

Source: Based on Problem 2, 8th Grade, National Final, László Kalmár National Mathematics Competition, Hungary, 2016. See

`https://matek.fazekas.hu/index.php?option=com_content&view=article&id=442:kalmar-laszlo-matematikaverseny&catid=26&Itemid=185`

Solution: First we will show that such placement of numbers is impossible if n is odd. Label the n points around the circle as P_k, where n is any integer and $P_k = P_l$ if $k \equiv l \mod n$, and such that P_k and P_{k+1} are adjacent. Let $f(P_k)$ be the integer placed at the point P_k. The first observation we can make is that we cannot have $f(P_k) = f(P_l)$ unless $P_k = P_l$. Indeed, if P_k and P_l are adjacent, say $l = k+1$, and $f(P_k) = f(P_l)$, then we have either $f(P_{k+1}) \mid f(P_{k-1})$ (i.e., $f(P_{k+1})$ is a divisor of $f(P_{k-1})$) or $f(P_{k-1}) \mid f(P_{k+1})$, neither of which is allowed. If P_k and P_l are not adjacent, then we have $f(P_k) \mid f(P_l)$, since $f(P_k) = f(P_l)$; this is not allowed, either.

Further, we cannot have $f(P_k) < f(P_{k+1}) < f(P_{k+2})$. Indeed, if this were the case, we would have $f(P_k) \mid f(P_{k+2})$, which is not allowed. Similarly, we cannot have $f(P_k) > f(P_{k+1}) > f(P_{k+2})$. Let $g(P_k) = 0$ if $f(P_k) < f(P_{k+1})$ and $g(P_k) = 1$ if $f(P_k) > f(P_{k+1})$. Then $g(P_k) \neq g(P_{k+1})$. This is not possible if n is odd (since $P_k = P_{k+n}$ and so $g(P_k) = g(P_{k+n})$). This shows that there is no placement of numbers around the circle if n is odd.

If n is even, an appropriate placement of numbers can be easily described. Writing $n = 2m$, let $p_1, p_2, \ldots, p_m$ be m distinct prime numbers. For i with $1 \leq i \leq n$ let $f(P_{2i}) = p_i$; this definite $f(P_l)$ for all even l, since $P_k = P_l$ if $k \equiv l \mod n$. For k odd, put $f(P_k) = f(P_{k-1})f(P_{k+1})$. This choice of f clearly satisfies the requirements.

8) (SENIOR 4) Given an equilateral triangle ABC, let P be an arbitrary point on the circle circumscribed to the triangle. Show that $PA^2 + PB^2 + PC^2$ is constant.

Source: Problem 1132, p. 78, Középiskolai Matematikai Lapok, Vol. XXIII, No. 2, October 1961,

`http://db.komal.hu/scan/1961/10/96110078.g4.png`

First solution: We may assume that the point P is on the arc between $\overset{\frown}{AB}$. Let $\alpha = \angle POA$, where A is the center of the circumscribed circle. Then $\angle POB = 2\pi/3 - \alpha$ and $\angle POC = 2\pi/3 + \alpha$. Writing r for the radius of the circle, and using the law of cosines for the triangles POA, POB, and POC, we obtain

$$\begin{aligned} PA^2 &= 2r^2(1 - \cos\alpha) \\ PB^2 &= 2r^2\left(1 - \cos\left(\frac{2\pi}{3} - \alpha\right)\right) \\ PC^2 &= 2r^2\left(1 - \cos\left(\frac{2\pi}{3} + \alpha\right)\right). \end{aligned}$$

Using the identity

$$\cos(x - y) + \cos(x + y) = 2\cos x \cos y,$$

which is an easy consequence of the addition formula for cosine, we obtain

$$\cos\left(\frac{2\pi}{3} - \alpha\right)\cos\left(\frac{2\pi}{3} + \alpha\right) = 2\cos\left(\frac{4\pi}{3}\right)\cos\alpha = -\cos\alpha,$$

where the last equation follows since $\cos(4\pi/3) = -1/2$. Hence, adding the equations for PA^2, PB^2, and PC^2, we obtain

$$PA^2 + PB^2 + PC^2 = 6r^2,$$

completing the proof that this sum is constant.

Second solution: The assertion generalizes to an arbitrary regular polygon. The easiest way to show this is by the use of complex numbers. Let $n \geq 2$ be an arbitrary integer.[1] Let $\zeta = e^{2k\pi i/n}$. Then the numbers ζ^k for $0 \leq k < n$ represent the vertices of a regular n-gon inscribed into the unit circle. In fact ζ is a root of the equation $z^n - 1 = 0$. Let z represent an arbitrary point on the unit circle; that is $|z| = 1$. Writing $\bar{w}$ for the conjugate of the complex number w, we have

$$|w|^2 = w\bar{w}.$$

We need to show that

$$\sum_{k=0}^{n-1} |z - \zeta^k|^2$$

is constant. Note that $\bar{z} = z^{-1}$, since $1 = |z|^2 = z\bar{z}$. Similarly, $\overline{\zeta^k} = \zeta^{-k}$. Hence

$$\begin{aligned}\sum_{k=0}^{n-1} |z - \zeta^k|^2 &= \sum_{k=0}^{n-1} (z - \zeta^k)(\bar{z} - \overline{\zeta^k}) = \sum_{k=0}^{n-1} (z - \zeta^k)(\bar{z}^{-1} - \zeta^{-k}) \\ &= \sum_{k=0}^{n-1} (1 - z^{-1}\zeta^k - z\zeta^{-k} + 1) = 2n + z^{-1}\sum_{k=0}^{n-1} \zeta^k + z\sum_{k=0}^{n-1} \zeta^{-k} = 2n;\end{aligned}$$

The last equation follows since the last two sums are zero. Indeed, we have

$$0 = \zeta^n - 1 = (\zeta - 1)\sum_{k=0}^{n-1} \zeta^k.$$

Since the first factor on the right-hand side is not zero, the second factor must be zero, that is

$$\sum_{k=0}^{n-1} \zeta^k = 0.$$

Taking conjugates, we also obtain that

$$\sum_{k=0}^{n-1} \zeta^{-k} = 0.$$

9) (SENIOR 5) Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 1$. Show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 5.$$

[1] There is no polygon of two sides (or else, it is a straight line segment, with the two sides overlapping each other). The case $n = 2$ of what follows is also true, as a simple consequence of the Pythagorean theorem and Thales's theorem. In what follows we do not need to exclude the case $n = 2$.

Source: Based on Problem 1417, p. 156, Középiskolai Matematikai Lapok, Vol. 31, No. 3–4, November 1965,

`http://db.komal.hu/scan/1965/11/96511156.g4.png`

Solution: The arithmetic, geometric, and harmonic means of the positive numbers a_1, a_2, ..., a_n are defined as

$$A = \frac{1}{n}\sum_{k=1}^{n} a_k, \qquad G = \left(\prod_{k=1}^{n} a_k\right)^{1/n}, \qquad \text{and} \qquad H = \frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}},$$

respectively. It is known that $A \ge G \ge H$, and equality holds only if all of the a_k are equal. The first of these inequalities is well known.[2] The second one is perhaps not so well known; it, however, easily follows from the inequality analogous to the first one between the arithmetic and geometric means of the numbers $1/a_k$. Using the inequalities between the various means, we obtain

$$\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \le \sqrt[3]{abc} = \sqrt{\sqrt[3]{a^2b^2c^2}} \le \sqrt{\frac{a^2+b^2+c^3}{3}} = \sqrt{\frac{1}{3}},$$

where the last equality holds since $a^2+b^2+c^2=1$. Taking reciprocals, this implies that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \sqrt{27} > \sqrt{25} = 5,$$

establishing the desired inequality.

10) (SENIOR 6) Let (a,b) be an interval on the real line, and let $c \in (a,b)$. Assume the real valued function f is continuous on (a,b), and differentiable at every point x in (a,b) that is different from c. Assume, further, that the limit $d = \lim_{x\to c} f'(x)$ exists (in particular, it is finite). Show that then f is also differentiable at c, and $f'(c) = d$.

Source: Given as Problem 2 on the Basic Ph.D. Qualifying Exam in mathematics, Fall 2007 at the University of California Los Angeles, but the result is well known, though rarely stated explicitly. See

`https://secure.math.ucla.edu/gradquals/hbquals.php`

Solution: Let $x \in (a,b)\setminus\{c\}$. By the Mean Value Theorem of differentiation, there is a ξ_x strictly between x and c (that is, if $x<c$ then $x<\xi_x<c$, and if $x>c$ then $x>\xi_x>c$) such that

$$\frac{f(x)-f(c)}{x-c} = f'(\xi_x).$$

[2]The easiest way to show the inequality between the arithmetic and geometric means is to note that the function $\log x$ (the natural logarithm of x – the symbol $\ln x$ for natural logarithm is rarely used in mathematics) is *concave* (that is, concave down in the terminology used in introductory calculus courses). Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\log a_k \le \log\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right),$$

since any chord of the graph of the logarithm lies below the curve. The name of the analogous inequality for an arbitrary concave function – or, more commonly, the reverse inequality for an arbitrary convex (i.e., concave up) function – is called *Jensen's inequality*.

Making $x \to c$, the number ξ_x will approach c, so $f'(\xi_x)$ will approach $d = \lim_{x\to c} f'(x)$. Hence, it follows that
$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} f'(\xi_x) = \lim_{x\to c} f'(x) = d.$$

Note 1: The way this proof is formulated, one defines a function that maps x to ξ_x. Since the choice of ξ_x is not unique, the definition of such a function involves the Axiom of Choice. Since the the Axiom of Choice is commonly used in arguments of introductory analysis,[3] one might not be worried about its use here. It is however easy to rewrite this proof in a way that requires no reference to the Axiom of Choice if one uses the Cauchy definition, i.e., the ϵ–δ definition of limit.[4]

Here is a proof that clearly avoids the Axiom of Choice. Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ such that $\delta \le \min(c-a, b-c)$ and such that for every ξ with $0 < |\xi - c| < \delta$ we have $|f'(\xi) - d| < \epsilon$. Let x be such that $0 < |x - c| < \delta$. By the Mean Value Theorem of differentiation, there is a ξ strictly between x and c (that is, if $x < c$ then $x < \xi < c$, and if $x > c$ then $x > \xi > c$) such that
$$\frac{f(x)-f(c)}{x-c} = f'(\xi).$$
Then we also have $0 < |\xi - c| < \delta$, and so
$$\left|\frac{f(x)-f(c)}{x-c} - d\right| = |f'(\xi) - d| < \epsilon.$$
Since $\epsilon > 0$ was arbitrary, this shows that
$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = d.$$
Perhaps the first proof we gave has greater intuitive appeal, but, as we pointed out, it ignores important foundational issues. Close attention to foundational issues is, however, uncommon even in rigorous introductions to analysis.

Note 1: If one wants to differentiate the function
$$f(x) = \frac{(x-1)^7(x+2)^9}{(x^2+4)^6},$$
one may simplify the calculation by using logarithmic differentiation, i.e., by calculating the derivative of $\log|f(x)|$ and then finding $f'(x)$ with the aid of the formula
$$\log|f(x)| = \frac{f'(x)}{f(x)};$$

[3]Actually, most uses of the Axiom of Choice can be replaced by the use of its significant weakening, called the Axiom of Dependent Choice, in introductory analysis. To show, however, that there is a set of reals that is not Lebesgue measurable the Axiom of Dependent Choice is not sufficient, according to a result of Robert M. Solovay. There are deep foundational issues we are skirting here that would require a much more extended discussion. Suffice it to say here that Solovay assumed the existence of an inaccessible cardinal in his proof, a natural assumption but unprovable in standard axiomatic set theory. Much later, Saharon Shelah showed that Solovay's assumption is indispensable in establishing the result.

[4]The Heine definition involving sequences is more problematic. In any case, one needs the Axiom of (Dependent) Choice to establish the equivalence of the Cauchy and Heine definitions.

here $\log t$ denotes the natural logarithm of t.[5] One may, however, wonder whether the result is correct also for $x = 1$ or $x = -2$, since $\log|f(x)|$ is not defined at these points. One may use the result of the present problem to confirm that the result is indeed correct also at these points.

11) (SENIOR 7) Denoting by $\mathbb{R}$ the set of real numbers, assume the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f(f(x)) = -x$$

for all real x. Show that f cannot be continuous everywhere.

Source: Peter M. Higgins, *Professor Higgins's Problem Collection*, Oxford University Press, New York, 2017.

Solution: First, observe that f must be one-to-one. In fact, if $f(x_1) = f(x_2)$, then

$$x_1 = -f(f(x_1)) = -f(f(x_2) = x_2.$$

Note that $f(0) = 0$. Indeed, assume that $f(0) = a \neq 0$. Then $f(a) = f(f(0)) = -0 = 0$. Further, $f(f(a)) = -a$, yet $f(f(a)) = f(0) = a$, contradicting the fact that $f \circ f$ is a (single-valued) function.

A set of reals S is called connected if there are no open sets U and V of reals such that that $(S \cap U) \cup (S \cap V) = S$ and the sets $S \cap U$ and $S \cap V$ are nonempty and disjoint (i.e., their intersection is empty). It is well known that the only nonempty connected subsets of the real line are intervals (open, closed, or semiclosed, including the degenerate closed intervals whose endpoints are the same, i.e., that are sets containing only one point.). It is a basic fact about continuous functions that if the set $S \subset \mathbb{R}$ is connected and g is continuous on $\mathbb{R}$, then the set $f[S] \stackrel{def}{=} \{f(x) : x \in S\}$ is also connected.

Since both the intervals $[0, +\infty)$ and $(0, +\infty)$ are connected, the image sets $f[[0, +\infty)]$ and $f[(0, +\infty])$ are both intervals. As f is one-to-one, the only difference between them is that the former contains $f(0) = 0$ and the latter does not. The only way this is possible is that 0 is an endpoint of both intervals. Similarly, the image set $f[(-\infty, 0)]$ is also an interval with one of its endpoint being 0. Note that the argument shows that neither of the intervals $f[(0, +\infty])$ and $f[(-\infty, 0)]$ contains the endpoint 0; further, these intervals must be disjoint, since f is one-to-one. In other words, there are two possibilities: (i) if $x > 0$ then $f(x) > 0$ and if $x < 0$ then $f(x) < 0$, or (i) if $x > 0$ then $f(x) < 0$ and if $x < 0$ then $f(x) > 0$. Both these possibilities contradict the equation $f(f(x)) = -x$; indeed, if $x > 0$, both possibilities imply that $f(f(x)) > 0$.

It is not too difficult to reformulate this proof in terms of the Intermediate Value Theorem for continuous functions, but the argument involving connectedness is more direct; in any case, the simplest proof of the Intermediate Value Theorem uses connectedness.

Note: A function on $\mathbb{R} \setminus \{-1, 1\}$ satisfying the equation $f(f(x)) = -x$ for all x in its domain can be defined as follows.[6] Put $f(0) = 0$, $f(x) = 1/x$ if $|x| < 1$ and $x \neq 0$, and $f(x) = -1/x$ if $|x| > 1$. To define such an f on the whole real line, keep the same definition unless $x = \pm 2^n$ for some integer n (positive, negative, or zero). If n is even, put $f(2^n) = 2^{n+1}$ and $f(-2^n) = -2^{n+1}$, and if n is odd, put $f(2^n) = -2^{n-1}$ and $f(-2^n) = 2^{n-1}$.

[5] This is the usual notation in mathematical writing. The symbol $\ln t$ is rarely used in mathematics outside of introductory calculus courses.

[6] The set $\{-1, 1\}$ denotes the set containing only the numbers -1 and 1.