Junior Prize Exam
Spring 2019

1) Find all pairs of positive integers $(n, k)$ such that

$$n! + 8 = 2^k.$$

2) Let $x$ and $y$ be integers. Show that

$$x(x+1) \neq 2(5y+2).$$

3) Given positive real numbers $a$, $b$, and $c$, prove that

$$(a+b)(b+c)(c+a) \geq 8abc.$$

4) Given an isosceles trapezoid, consider a triangle with sides equal to the leg, the diagonal, and the geometric mean of the parallel sides of the trapezoid. Show that this triangle is a right triangle.

In other words, given a trapezoid $ABCD$ with $AB \parallel CD$, $AD = BC$, and $AD \nparallel BC$ unless $\angle DAB$ is a right angle, show that a triangle with sides of lengths $AD$, $BD$, and $\sqrt{AB \cdot CD}$ is a right triangle.

5) The function $f$ defined everywhere on the real line satisfies

$$f(x+1) = \frac{1+f(x)}{1-f(x)}$$

for all real $x$. Show that $f$ is periodic; that is, there is a positive real number $T$ such that $f(x+T) = f(x)$ for all real $x$.

6) Let $x$ be a nonzero real number such that

$$n = x + \frac{1}{x}$$

is an integer. Show that

$$A = x^4 + x^3 + x^2 + x^{-2} + x^{-3} + x^{-4}$$

is also an integer.

7) Let $n \geq 3$ be an integer. One wants to place $n$ positive integers around a circle in such a way that if $a$ and $b$ are adjacent then either $a$ is a divisor of $b$ or $b$ is a divisor of $a$, and if $a$ and $b$ are not adjacent then neither is the divisor of the other. Show that this is possible if and only if $n$ is even.

Soon after the exam, solutions will appear on the Web Site
http://www.sci.brooklyn.cuny.edu/~mate/prize/2019/

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All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. AMS-TeX was used for typesetting.