Senior Prize Exam
Spring 2019

1) Find all pairs of positive integers (n, k) such that

$$n! + 8 = 2^k.$$

2) Let x and y be integers. Show that

$$x(x + 1) \neq 2(5y + 2).$$

3) Given positive real numbers a, b, and c, prove that

$$(a + b)(b + c)(c + a) \geq 8abc.$$

4) Given an equilateral triangle ABC, let P be an arbitrary point on the circle circumscribed to the triangle. Show that $PA^2 + PB^2 + PC^2$ is constant.

5) Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 1$. Show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 5.$$

6) Let (a, b) be an interval on the real line, and let $c \in (a, b)$. Assume the real valued function f is continuous on (a, b), and differentiable at every point x in (a, b) that is different from c. Assume, further, that the limit $d = \lim_{x \to c} f'(x)$ exists (in particular, it is finite). Show that then f is also differentiable at c, and $f'(c) = d$.

7) Denoting by $\mathbb{R}$ the set of real numbers, assume the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f(f(x)) = -x$$

for all real x. Show that f cannot be continuous everywhere.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2019/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.