All Problems on the Prize Exams
Spring 2020
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Prove that the sum of the squares of five consecutive integers is divisible by 5, but it is not divisible by 25. Dániel Arany Mathematics Competition, Category 1, Round 1, 10th grade, Problem 3.

Source:

`http://versenyvizsga.hu/external/vvszuro/vvszuro.php`

Solution: Let the five integers be $n-2$, $n-1$, n, $n+1$, and $n+2$. Then

$$(n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 = 5n^2 + 10 = 5(n^2+2),$$

and this is clearly divisible by 5. To show that it is not divisible by 25, we need to show that n^2+2 is not divisible by 5. For this, we need to examine the cases $n = 5k$, $n = 5k \pm 1$, and $n = 5k \pm 2$ for some integer k. In these cases, $n^2 + 2$ in turn equals $25k^2 + 2$, $25k^2 \pm 10k + 3$, $25k^2 \pm 20k + 6$. It is clear that none of these numbers are divisible by 5.

2) (JUNIOR 2 and SENIOR 2) Let n be an integer and let $x = 3n - 1$. Show that

$$x^6 - x^3 - x^2 + x$$

is divisible by 9.

Source: Based on Problem 897, p. 140, Középiskolai Matematikai Lapok, Vol. VIII, No. 6, January 1901,

`http://db.komal.hu/scan/1901/01/90101140.g4.png`

Solution: Using the binomial theorem, for a positive integer $k \ge 1$, we have

$$\begin{aligned} x^k &= (3n-1)^k = \big(3n + (-1)\big)^k \\ &= \ldots + \binom{k}{1} 3n(-1)^{k-1} + \binom{k}{k}(-1)^k = \ldots + (-1)^{k-1}(3kn - 1), \end{aligned}$$

where the terms indicated by $\ldots$ are terms are divisible by 9 (since they contain powers higher than 1 of $3n$, or, in case $k = 1$, they are nonexistent). Thus,

$$x^k \equiv (-1)^{k-1}(3kn - 1) \mod 9 \qquad (k \ge 1);$$

therefore

$$\begin{aligned} x^6 - x^3 - x^2 + x &\equiv -(3\cdot 6n - 1) - (3 \cdot 3n - 1) - \big(-(3\cdot 2n - 1)\big) + (3n - 1) \\ &= (-18 - 9 + 6 + 3)n + (1 + 1 - 1 - 1) \equiv 9n \equiv 0 \mod 9, \end{aligned}$$

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

which is what we wanted to prove.

3) (JUNIOR 3 and SENIOR 3) Let $A \neq 0$, $B \neq 0$, $C \neq 0$ and a, b, c be real numbers, and assume that

$$a+b+c=0, \qquad A+B+C=0, \qquad \frac{a}{A}+\frac{b}{B}+\frac{c}{C}=0.$$

Show that

$$aA^2+bB^2+cC^2=0.$$

Source: Dániel Arany Mathematics Competition, beginners' level (9th grade), round 2, 1954. See Középiskolai Matematikai Lapok, No. 10, October 1954,

`http://db.komal.hu/scan/1954/10/95410036.g4.png`

Solution: Writing

$$S=aA^2+bB^2+cC^2,$$

and noting that we have $A=-B-C$, $B=-A-C$, and $C=-A-B$ according to the second of the given equations, we obtain that

$$\begin{aligned}2S=S+S&=\left(aA^2+bB^2+cC^2\right)+\left(a(B+C)^2+b(A+C)^2+c(A+B)^2\right)\\&=a(A^2+B^2+C^2)+b(A^2+B^2+C^2)+c(A^2+B^2+C^2)+2aBC+2bAC+2cAB\\&=(a+b+c)(A^2+B^2+C^2)+2(aBC+bAC+cAB).\end{aligned}$$

The first term on the right-hand side is 0 in view of first of the given equations, and the second term is 0 as can be seen by multiplying the third of the given equations by ABC. Hence $S=0$, as we wanted to show.

4) (JUNIOR 4) Given real numbers $x_i \in [0,1]$ for $1 \le i \le n$, show that

$$\left(1+\sum_{i=1}^{n} x_i\right)^2 \ge 4\sum_{i=1}^{n} x_i^2.$$

Source: Problem 278, Problems of the All-Soviet-Union math competitions 1961-1986,
`http://web.archive.org/web/20120825124642/http://pertselv.tripod.com/RusMath.html`

Solution: Using the identity $a^2-b^2=(a+b)(a-b)$, we have

$$\left(1+\sum_{i=1}^{n} x_i\right)^2-\left(-1+\sum_{i=1}^{n} x_i\right)^2=2\sum_{i=1}^{n} x_i\cdot 2=4\sum_{i=1}^{n} x_i.$$

Hence

$$\left(1+\sum_{i=1}^{n} x_i\right)^2=4\sum_{i=1}^{n} x_i+\left(-1+\sum_{i=1}^{n} x_i\right)^2\ge 4\sum_{i=1}^{n} x_i \ge 4\sum_{i=1}^{n} x_i^2;$$

the last inequality holds since $x_i \ge x_i^2$, given that $0 \le x_i \le 1$. This establishes the inequality in question.

5) (JUNIOR 5) How many ways can one place 2 red, 3 green, and 4 blue balls in a row in such a way that no red ball is placed next to a green ball. (We count color arrangements; that is, two balls of the same color are indistinguishable.)

Source: Problem 461, p. 156, Középiskolai Matematikai Lapok, Vol. IV, No. 4–5, May–June, 1952. See

`http://db.komal.hu/scan/1952/05/95205156.g4.png`

Solution: We will study the different ways of placing the red balls.

1. Placing the red balls in such a way that fixes the placement of one blue ball. There are two such placement, in which the two red balls are placed next to each other at either end. In these arrangements of one blue ball must be placed at the free place next to one of the red balls (so no green ball will be placed there). With each arrangement, the remaining balls (3 blue balls and 3 green balls) can be arranged in $\binom{6}{3} = 20$ ways, giving rise to $2 \cdot 20 = 40$ arrangements.

2. Placing the red balls in such a way that fixes the placements of two blue balls. There are $1 + 2 + 6$ such ways: placing a red ball at each end, or one red ball at one end, leaving a gap of one place, and placing the other red ball, or placing the two red balls next to each other, but not at the end. Each of these 9 arrangements leaves the placement of the remaining 2 blue balls and 3 green balls to be arranged in $\binom{5}{2} = 10$ ways, resulting in a total of $6 \cdot 10 = 90$ arrangements.

3. Placing the red balls in such a way that fixes the placements of three blue balls. There are $2 \cdot 6 + 5 = 17$ such ways: placing a red ball at one end, leaving a gap of at least two places, then placing the other red ball anywhere but not at the end, or placing the two red balls with exactly one gap between them, but placing neither ball at the end. The remaining 1 blue ball and 3 green balls can be placed in $\binom{4}{1} = 4$ ways, resulting in $17 \cdot 4 = 68$ arrangements.

4. Placing the red balls in such a way that fixes the placements of all four blue balls. In these arrangements, there is a gap of size at least two between the red balls, and neither red ball is placed at the end. There are $4 + 3 + 2 + 1 = 10$ such placements (if the red ball on the left is put at place 2, the other red ball can be put at places 4, 5, 6, or 7, if it is put at place 3, the other red ball can be put at places 5, 6, or 7, etc.). With each of these placements, the remaining green balls can only be placed in one way, resulting in a total or 10 arrangements.

Adding up the number of arrangements given in each of the cases, we obtain a total of $40 + 90 + 68 + 10 = 208$ arrangements.

6) (JUNIOR 6) Find all integer solutions of the system of equations

$$x + y + z + t = 22, \qquad xyzt = 648,$$
$$\frac{1}{x} + \frac{1}{y} = \frac{7}{12}, \qquad \frac{1}{z} + \frac{1}{t} = \frac{5}{18}.$$

Source: Based on Problem 2040, p. 175, Középiskolai Matematikai Lapok, Vol. XVIII, No. 8, March 1911

`http://db.komal.hu/scan/1911/03/91103175.g4.png`

Solution: It is easy to solve the system of equations

$$u + v = p, \qquad uv = q$$

for the unknowns u and v. Indeed, assuming these equations are satisfied, we have

$$(\zeta - u)(\zeta - v) = \zeta^2 - (u + v)\zeta + uv = \zeta^2 - p\zeta + q.$$

Since the left-hand side is 0 exactly when $\zeta = u$ or $\zeta = v$. That is, u and v can be found as the two solutions of the equation

$$\zeta^2 - p\zeta + q = 0$$

for ζ.

Going back to the original system of equations, the third and the fourth equations can be written as

$$(1) \qquad x + y = \frac{7}{12}xy \qquad \text{and} \qquad z + t = \frac{5}{18}zt.$$

Hence, writing

$$u = x + y \qquad \text{and} \qquad v = z + t,$$

the first and second equations can be written as

$$u + v = x + y + z + t = 22$$

and

$$uv = \frac{7}{12}xy \cdot \frac{5}{18}zt = \frac{7}{12} \cdot \frac{5}{18}xyzt = \frac{7}{12} \cdot \frac{5}{18} \cdot 648 = 105.$$

That is

$$u + v = 22 \qquad \text{and} \qquad uv = 105.$$

The solutions of the equations $\zeta^2 - 22\zeta + 105 = 0$ are 7 and 15. Therefore, we have

$$u = \frac{7}{12}xy = 7 \quad \text{and} \quad v = \frac{5}{18}zt = 15 \qquad \text{or} \qquad u = \frac{7}{12}xy = 15 \quad \text{and} \quad v = \frac{5}{18}zt = 7.$$

The second choice does not give integer values for xy, so we need to pick the first choice. So we must have $xy = 12$, and so $x+y = 7$ by the first equation in (1). Similarly, we must have $zt = 3 \cdot 18 = 54$, and so $z + t = 15$ by the second equations in (1). That is, x and y can be obtained as the solutions of the equation $\zeta^2 - 7\zeta + 12 = 0$, i.e., $x = 3$ and $y = 4$ or $x = 4$ or $y = 4$. Similarly, z and t can be obtained as the solutions of the equation $\zeta^2 - 15\zeta + 54 = 0$, i.e., $z = 9$ and $t = 6$ or $z = 6$ and $t = 9$. These give all integer solutions of the above system of equations, That is, (x, y, z, t) can be

$$(3, 4, 6, 9) \quad \text{or} \quad (4, 3, 6, 9) \quad \text{or} \quad (3, 4, 9, 6) \quad \text{or} \quad (4, 3, 9, 6).$$

The equations have four more solutions that involve fractions; they can be found by taking the second choice for u and v.

7) (JUNIOR 7) Let $P(x)$ be a polynomials with real coefficients, and assume that there is no polynomial $Q(x)$ such that $P(x) = \big(Q(x)\big)^2$. Show that there is no polynomial $R(x)$ such that $P(P(x)) = \big(R(x)\big)^2$.

Source: Problem 3 (simplified), József Kürschák Mathematical Competition, Hungary, 2016. See

`http://www.ematlap.hu/index.php/hirek-ujdonsagok-2017-03/`
`420-jelentes-a-2016-evi-kurschak-jozsef-matematikai-tanuloversenyrol`

Solution: Let

$$(1) \qquad P(x) = a\prod_{j=1}^{n}(x - \alpha_j)^{k_j},$$

where the α_j's are pairwise distinct complex numbers, and $k_j \geq 1$ is the multiplicity of the zero α_j. Given that $P(x)$ is not the square of a polynomial, there must be at least one among the k_j's that is odd. We may assume that k_1 is odd. We have

$$P(P(x)) = a \prod_{j=1}^{n} \big(P(x) - \alpha_j\big)^{k_j}.$$

Since the factors $P(x) - \alpha_j$ are all relatively prime to each other (i.e., no two of them has a common factor of degree ≥ 1, since the difference of any two is a nonzero constant), for $P(P(x))$ to be a square of a polynomial, each of the factors with an odd exponent must itself be a square of a polynomial.

In order for $P(x) - a_1$ to be a square of a polynomial, it must have an even degree. That is, the degree of $P(x)$ must be even. For this, there must be an exponent k_j other than k_1 that is odd in the factorization (1); we may assume that k_2 is odd.

It is, however, not possible for there to be polynomials $T_1(x)$ and $T_2(x)$ such that $P(x) - \alpha_1 = \big(T_1(x)\big)^2$ and $P(x) - \alpha_2 = \big(T_2(x)\big)^2$. Indeed, were this the case, we would have

$$\alpha_2 - \alpha_1 = \big(P(x) - \alpha_1\big) - \big(P(x) - \alpha_2\big) = \big(T_1(x)\big)^2 - \big(T_2(x)\big)^2 = \big(T_1(x) + T_2(x)\big)\big(T_1(x) - T_2(x)\big).$$

Here, the left-hand side is not zero and has degree 0, while the right-hand side has degree at least 1, so this equation cannot hold. This shows that $P(P(x))$ cannot be a square of a polynomial under the given assumptions.

Note. We simplified the problem in that the problem asks whether or not $P(P(x))$ can be a square assuming that no polynomial $Q(x)$ exists as described. Our proof uses the Fundamental Theorem of Algebra, which guarantees the existence of the factorization (1). The above website mentions that there is an elegant solution of the problem that does not use the Fundamental Theorem of Algebra. No solution is described at the website.

8) (SENIOR 4) Let x and y be two positive reals. Prove that

$$xy \leq \frac{x^{n+2} + y^{n+2}}{x^n + y^n} \qquad (n \geq 0).$$

Source: Problem 1, Pan African Mathematics Competition, Day 2, 2008. See
`https://artofproblemsolving.com/community/c4521_2008_pan_african`

Solution: Without loss of generality we may assume that $x \leq y$. Dividing the inequality to be proved by x^2, and then dividing the numerator and denominator on the right-hand side by x^n, the inequality can be written as

$$\frac{y}{x} \leq \frac{1 + \left(\frac{y}{x}\right)^{n+2}}{1 + \left(\frac{y}{x}\right)^{n}}.$$

Writing $t = y/x$, we have $t \geq 1$ since we assumed $x \leq y$, and the inequality becomes

$$t \leq \frac{1 + t^{n+2}}{1 + t^n} \qquad (t \geq 1).$$

Multiplying both sides by $1 + t^n$, we obtain

$$t^{n+1} + t \leq t^{n+2} + 1 \qquad (t \geq 1).$$

After rearranging this, we obtain the inequality

$$t^{n+1} - 1 \le t(t^{n+1} - 1), \qquad (t \ge 1),$$

which is equivalent to the inequality to be proved. Making use of the fact that the left-hand side is nonnegative since $n \ge 0$, this inequality is clearly true, completing the proof of the above inequality.

Note. The intention was that n is a positive integer, but this was missing from the statement of the problem. However, as the proof shows, the result is true for any real $n \ge 0$. The source does state that n is a nonnegative integer.

9) (SENIOR 5) Let n be a positive integer and

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$$

be a polynomial with complex coefficients. If r is a complex number such that $p(r) = 0$, show that

$$|r| \le \max\Big(1, \sum_{k=0}^{n-1} |a_k|\Big)$$

Source: Written Qualifying Examination, Fall 2001, Day 2, Problem 9, Department of Mathematics, Rutgers University. See

`https://www.math.rutgers.edu/academics/graduate-program/program-requirements/requirements-for-the-doctoral-degree/written-qualifying-exam`

Solution: There is nothing to prove if $|r| \le 1$, so we may assume that $|r| > 1$. We have

$$0 = \frac{p(r)}{r^{n-1}} = r + \sum_{k=0}^{n-1} \frac{a_k}{r^{n-1-k}}.$$

Hence

$$r = -\sum_{k=0}^{n-1} \frac{a_k}{r^{n-1-k}}.$$

Taking absolute values, noting that the exponent of r in the denominator is nonnegative, and using the assumption $|r| > 1$, we obtain

$$|r| = \left|\sum_{k=0}^{n-1} \frac{a_k}{r^{n-1-k}}\right| \le \sum_{k=0}^{n-1} \left|\frac{a_k}{r^{n-1-k}}\right| \le \sum_{k=0}^{n-1} |a_k|.$$

This completes the proof (note that the last inequality needs to hold only in case $|r| > 1$).

10) (SENIOR 6) Let f be a continuously differentiable function for $x \ge 0$, and assume it satisfies the equation

$$x = f(x)e^{f(x)}$$

for all $x \geq 0$. Calculate

$$\int_0^e f(x)\,dx.$$

Source: Problem 10, February 15, 2014, Calculus Test, Rice Mathematics Tournament. See `http://www.ruf.rice.edu/ eulers/prevtests.html`

Solution: We have $f(x) = xe^{-f(x)}$ according to the equation $x = f(x)e^{f(x)}$, and so, using integration by parts, we have

$$\int_0^e f(x)\,dx = \int_0^e xe^{-f(x)}\,dx = \frac{x^2}{2}e^{-f(x)}\Big|_{x=0}^{x=e} + \int_0^e \frac{x^2}{2}e^{-f(x)}f'(x)\,dx$$
$$= \frac{e^2}{2}e^{-f(e)} + \frac{1}{2}\int_0^e \big(f(x)\big)^2 e^{f(x)}f'(x)\,dx;$$

to obtain the second equation, using the equation $x = f(x)e^{f(x)}$, we replaced x^2 under the integral sign with $\big(f(x)\big)^2 e^{2f(x)}$. Substituting $t = f(x)$ in the integral on the right-hand side, we have $dt = f'(x)\,dx$; as for the limits, the lower limit will be 0, since $f(0) = 0$ according to the equation $x = f(x)e^{f(x)}$. After this substitution, the integral is easily evaluated by repeated integrations by parts. Hence we have

$$\int_0^e f(x)\,dx = \frac{e^2}{2}e^{-f(e)} + \frac{1}{2}\int_0^{f(e)} t^2e^t\,dt$$

Before proceeding any further, we will determine the value of $f(e)$. According to the equation $x = f(x)e^{f(x)}$, $x = f(e)$ is a solution of the equation $e = xe^x$, that is, of the equation $x = ee^{-x}$. The solution of this equation is unique; indeed, in the xy coordinate system, this solution can be obtained as the x-coordinate of the intersection of the curves $y = x$ and $y = ee^{-x}$; since the former curve is increasing, while the latter one is decreasing, there is only one point of intersection. Since $x = 1$ is obviously a solution, it follows that $f(e) = 1$.

Hence we have

$$\int_0^e f(x)\,dx = \frac{e^2}{2}e^{-1} + \frac{1}{2}\int_0^1 t^2e^t\,dt = \frac{e}{2} + \frac{1}{2}(t^2 - 2t + 2)e^t\Big|_{t=0}^{t=1}$$
$$= \frac{e}{2} + \frac{e}{2} - 1 = e - 1,$$

where we used repeated integrations by part to evaluate the integral.

Note. The argument used to determine $f(e)$ can also be used to show that equation $x = f(x)e^{f(x)}$ uniquely determines $f(x)$ for $x \geq 0$. Further, $f(x)$ can also be obtained as a solution of a differential equation, and then using standard theorems about differential equation it will follow that f is continuously differentiable for $x \geq 0$. Indeed, writing $y = f(x)$ and differentiating the equation $x = ye^y$, we obtain $1 = y'e^y + yy'e^y$, and so $y = f(x)$ is the solution of the differential equation

$$y' = \frac{e^{-y}}{1+y}.$$

Given an initial condition $f(0) = y_0 \geq 0$, this equation is uniquely solvable for all $x \geq 0$. The equation $x = f(x)e^{f(x)}$, specifies the initial condition $f(0) = 0$.

11) (SENIOR 7) Let U be a finite dimensional vector space over the complex numbers.

(i) Let $S, T : U \to U$ be linear transformations, and assume λ is an eigenvalue of ST. Show that λ is also an eigenvalue of TS (a complex number λ is an eigenvalue of a linear transformation T if $Tx = \lambda x$ for a nonzero vector x; x is called the eigenvector associated with the eigenvalue λ).

(ii) Writing $I : U \to U$ for the identity transformation, show that there are no linear transformations $T, S : U \to U$ such that $ST - TS = I$.

Source: Part (ii) is given as Problem 6c on Ph. D. Qualifying Examination given in January 2005 in Pure Mathematics at the University of British Columbia. See

`https://www.math.ubc.ca/Grad/QualifyingExams/`

Solution: Part (i). Let $x \in U$ be a nonzero vector such that $STx = \lambda x$. Then we have $TS(Tx) = \lambda(Tx)$. If $Tx \neq 0$, then this equation shows that λ is also an eigenvalue of TS with eigenvector Tx. Assume therefore that $Tx = 0$; in this case we also have $STx = 0$, so we must have $\lambda = 0$ according to the equation $STx = \lambda x$. We then have to show that 0 is also an eigenvalue of the transformation TS.

Writing $\ker T \overset{def}{=} \{v \in U : Tv = 0\}$ for the kernel, or null space, of T, we have $x \in \ker T$. given that $Tx = 0$. As $x \neq 0$, it follows that $\dim \ker T > 0$, where $\dim V$ for a vector space V denotes its dimension. By the rank-nullity theorem[1] we have

$$\dim U = \dim \ker T + \dim \operatorname{ra} T,$$

where $\operatorname{ra} T$ is the range of T. Hence $\dim \operatorname{ra} T < \dim U$. Since $\operatorname{ra} TS \subset \operatorname{ra} T$, it follows that $\dim \operatorname{ra} TS \leq \dim \operatorname{ra} T < \dim U$ and so, applying the rank-nullity theorem to TS, we can see that $\dim \ker TS > 0$. Hence, there is a nonzero vector $y \in U$ such that $TSy = 0$. This shows that 0 is indeed an eigenvalue of TS, as we wanted to show.

Part (ii). Assume $ST - TS = I$, and let λ be an eigenvalue of TS. If x is a corresponding eigenvector, then $STx = TSx + x = (\lambda + 1)x$, so $\lambda + 1$ is an eigenvalue of ST. By part (i) of this problem, then $\lambda + 1$ is also an eigenvalue of TS. Continuing this argument, we can see that $\lambda + k$ is an eigenvalue of TS for all integers $k \geq 0$. Since no transformation on a finite dimensional space can have infinitely many eigenvalues (as we will explain below), this is impossible, showing that no such S and T can exist.

Given a linear transformation $T : U \to U$, one can show that T cannot have more eigenvalues than $\dim U$ as follows. If λ_i for $1 \leq i \leq m$ are pairwise distinct eigenvalues of T and x_i is an eigenvector corresponding to the eigenvalue λ_i, then the system $(x_1, x_2, \ldots, x_m)$ is linearly independent (so we cannot have $m > \dim U$). Indeed, assume that we have

$$\sum_{i=1}^{m} \alpha_i x_i = 0,$$

where not all the numbers α_i are zero. Assume that the α_i's here are so chosen that the smallest possible number among them is not zero. Let the number of nonzero coefficients among the α_i

[1] For a discussion of the rank-nullity theorem, see my notes `jordan_canonical.pdf` at the site

`http://www.sci.brooklyn.cuny.edu/~mate/misc/`

on pp. 2–3 in Subsection 1.1. On p. 3 in Subsection 1.2, the following result, used below in the solution for part (ii), is established: If U is a finite-dimensional vector space over the complex numbers, and $R : U \to U$ is a linear transformation, then R has an eigenvalue.

be k, where $1 \le k \le m$. By rearranging the vectors and the corresponding eigenvalues, we may assume that $\alpha_i \ne 0$ for i with $1 \le i \le k$ and $\alpha_i = 0$ for $k < i \le m$, where $k \ge 1$. That is,

$$\sum_{i=1}^{k} \alpha_i x_i = 0.$$

Of course, $k = 1$ is not possible here, since it would mean that $x_1 = 0$, whereas we assumed that an eigenvector cannot be the zero vector. Applying the linear transformation T to the vector on the left of this equation, and noting that $Tx_i = \lambda_i x_i$, we obtain

$$\sum_{i=1}^{k} \alpha_i \lambda_i x_i = 0.$$

Multiplying the former linear relation by λ_k and subtracting from it the latter, we obtain

$$\sum_{i=1}^{k-1} (\alpha_i \lambda_k - \alpha_i \lambda_i) x_i = 0.$$

None of the coefficients $\alpha_i \lambda_k - \alpha_i \lambda_i$ $(1 \le i \le k-1)$ is zero, since $\lambda_i \ne \lambda_k$ by our assumption. This relation contradicts the minimality of k, showing that the system $(x_1, x_2, \dots, x_m)$ is indeed linearly independent.

Note. As linear transformations on finite dimensional spaces can be identified with matrices, the question could equally well have been asked for $n \times n$ matrices for a positive integer n. Part (i) is probably well known; it was asked mainly because it gives a hint for solving part (ii), although it is possible that a solution independent of part (i) can be given for part (ii). The fact that the number of distinct eigenvalues is finite is probably also well known, since the eigenvalues of an $n \times n$ matrix A are the zeros of its characteristic polynomial $\det(A - \lambda I)$, where I is the identity matrix. Finally, as for (i) formulated for matrices, if A and B are $n \times n$ matrices over a commutative ring with a unit element, then it can be shown that

$$\det(AB - \lambda I) = \det(BA - \lambda I) \tag{1}$$

is an identity. Indeed, if A and B are matrices over the reals and A is nonsingular, then we have

$$\begin{aligned} \det(AB - \lambda I) &= \det = \big(A(B - \lambda A^{-1})\big) = \det(A)\det(B - \lambda A^{-1}) \\ &= \det(B - \lambda A^{-1})\det(A) = \big((B - \lambda A^{-1})A\big) = \det(BA - \lambda I). \end{aligned}$$

This is enough to imply that (1) is an identity on all commutative rings with a unit element by the argument given in the the notes `cayley_hamilton.pdf` at the website

`http://www.sci.brooklyn.cuny.edu/~mate/misc/`

on pp. 1–2 in Section 1.