Junior Prize Exam
Spring 2020

1) Prove that the sum of the squares of five consecutive integers is divisible by 5, but it is not divisible by 25.

2) Let n be an integer and let $x = 3n - 1$. Show that

$$x^6 - x^3 - x^2 + x$$

is divisible by 9.

3) Let $A \neq 0$, $B \neq 0$, $C \neq 0$ and a, b, c be real numbers, and assume that

$$a + b + c = 0, \qquad A + B + C = 0, \qquad \frac{a}{A} + \frac{b}{B} + \frac{c}{C} = 0.$$

Show that

$$aA^2 + bB^2 + cC^2 = 0.$$

4) Given real numbers $x_i \in [0, 1]$ for $1 \leq i \leq n$, show that

$$\Bigl(1 + \sum_{i=1}^{n} x_i\Bigr)^2 \geq 4 \sum_{i=1}^{n} x_i^2.$$

5) How many ways can one place 2 red, 3 green, and 4 blue balls in a row in such a way that no red ball is placed next to a green ball. (We count color arrangements; that is, two balls of the same color are indistinguishable.)

6) Find all integer solutions of the system of equations

$$\begin{aligned} &x + y + z + t = 22, \qquad &&xyzt = 648, \\ &\frac{1}{x} + \frac{1}{y} = \frac{7}{12}, &&\frac{1}{z} + \frac{1}{t} = \frac{5}{18}. \end{aligned}$$

7) Let $P(x)$ be a polynomials with real coefficients, and assume that there is no polynomial $Q(x)$ such that $P(x) = \bigl(Q(x)\bigr)^2$. Show that there is no polynomial $R(x)$ such that $P(P(x)) = \bigl(R(x)\bigr)^2$.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2020/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.