SENIOR PRIZE EXAM
SPRING 2020

1) Prove that the sum of the squares of five consecutive integers is divisible by 5, but it is not divisible by 25.

2) Let n be an integer and let $x = 3n - 1$. Show that

$$x^6 - x^3 - x^2 + x$$

is divisible by 9.

3) Let $A \neq 0$, $B \neq 0$, $C \neq 0$ and a, b, c be real numbers, and assume that

$$a + b + c = 0, \qquad A + B + C = 0, \qquad \frac{a}{A} + \frac{b}{B} + \frac{c}{C} = 0.$$

Show that

$$aA^2 + bB^2 + cC^2 = 0.$$

4) Let x and y be two positive reals. Prove that

$$xy \leq \frac{x^{n+2} + y^{n+2}}{x^n + y^n} \qquad (n \geq 0).$$

5) Let n be a positive integer and

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$$

be a polynomial with complex coefficients. If r is a complex number such that $p(r) = 0$, show that

$$|r| \leq \max\Big(1, \sum_{k=0}^{n-1} |a_k|\Big)$$

6) Let f be a continuously differentiable function for $x \geq 0$, and assume it satisfies the equation

$$x = f(x)e^{f(x)}$$

for all $x \geq 0$. Calculate

$$\int_0^e f(x)\,dx.$$

7) Let U be a finite dimensional vector space over the complex numbers.

(i) Let $S, T : U \to U$ be linear transformations, and assume λ is an eigenvalue of ST. Show that λ is also an eigenvalue of TS (a complex number λ is an eigenvalue of a linear transformation T if $Tx = \lambda x$ for a nonzero vector x; x is called the eigenvector associated with the eigenvalue λ).

(ii) Writing $I : U \to U$ for the identity transformation, show that there are no linear transformations $T, S : U \to U$ such that $ST - TS = I$.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE
http://www.sci.brooklyn.cuny.edu/~mate/prize/2020/

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.