All Problems on the Prize Exams
Spring 2021 Version Date: Mon Mar 8 18:20:24 EST 2021

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. The Junior Prize Exam was not given this year.

1) (SENIOR 1) Find all positive integers x such that $x^5 - 3x^2 = 216$.

Source: Probl 10, p. 8, Középiskolai Matematikai Lapok, No. 00, Vol. 1 January 1994.

`http://db.komal.hu/scan/1894/00/89400008.g4.png`

Solution: We have $(x^3-3)x^2 = 216$. The prime factorization of 216 is $2^3 \cdot 3^3$. Since $x^3 - 3$ is not divisible by 3^2, we must have $3 \mid x^2$; so $x \geq 3$. Then $x^3 - 3 \geq 24$. Since $24 \cdot 9 = 216$, we cannot have $x > 3$; hence $x = 3$.

2) (SENIOR 2) If n is an integer, show that

$$\frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$

is also an integer.

Source: Problem 1034, p. 203, Középiskolai Matematikai Lapok, Vol. IX/9, April 1902. See

`http://db.komal.hu/scan/1902/04/90204203.g4.png`

Solution: We need to show that

$$6\left(\frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}\right) = n + 3n^2 + 2n^3$$

is divisible by 6. This can be accomplished by showing that it is divisible by both 2 and 3.

As for divisibility by 2, we have $n + 3n^2 + 2n^3 = n(n+1)(2n+1)$, and here $n(n+1)$ is always divisible by 2, since one of n and $n+1$ is even. As for divisibility by 3, we have

$$n + 3n^2 + 2n^3 = 3(n^2+n^3) + n - n^3 = 3(n^2+n^3) - n(n+1)(n-1).$$

Since one of the numbers n, $n-1$, and $n+1$ is divisible by 3, both terms here are divisible by 3.

3) (SENIOR 3) Let n be a positive integer. Assume we are given n (not necessarily distinct) integers such that their sum is 0 and their product is n itself. Prove that n is divisible by 4.

Source: Problem 371, The problems of the All-Soviet-Union mathematical competitions 1961-1986. See

`http://web.archive.org/web/20120825124642/http://pertselv.tripod.com/RusMath.html`

Solution: The number of odd numbers among the given numbers must be even, otherwise the sum would not be 0. There must be an even one among the numbers; otherwise the product would not be even, and the number n of the given numbers would be even. Given that there is an even number, the product is even, the the number n of the given numbers must also be even. Since number of odd ones among these is even, so must be the number of the even ones, so that the total

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

number is even. So, the product n of the given numbers is divisible by 4, since there are at least two even ones among them.

4) (SENIOR 4) Show that for every integer $n > 5$ we have

$$\left(\frac{n}{2}\right)^n > n!.$$

Source: Problem 1595, p. 173, Középiskolai Matematikai Lapok, Vol. XIV/3, March 1907. See `http://db.komal.hu/scan/1907/03/90703173.g4.png`

First solution: Observe that the function $f(x) = 4x(1-x)$ assumes is increasing if $x \le 1/2$, decreasing if $x > 1/2$. It has maximum at $x = 1/2$, when it assumes the value 1. Hence, assuming that $n \ge 6$ and k with $1 \le k \le n-1$, we have

$$\frac{4k(n-k)}{n^2} = 4\frac{k}{n}\left(1 - \frac{k}{n}\right) = f\left(\frac{k}{n}\right) \le 1.$$

For the values $k = 1$ and 2,, we have better estimates. For $k = 1$, we have

$$f\left(\frac{1}{n}\right) = f\left(\frac{n-1}{n}\right) \le f\left(\frac{1}{6}\right) = \frac{5}{9}.$$

For $k = 2$, we have

$$f\left(\frac{2}{n}\right) = f\left(\frac{n-2}{n}\right) \le f\left(\frac{2}{6}\right) = \frac{8}{9}.$$

Hence

$$f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) = f\left(\frac{n-2}{n}\right) f\left(\frac{n-1}{n}\right) = \frac{40}{81} < \frac{1}{2}.$$

Hence, for $n \ge 6$ we have

$$\begin{aligned}
\left(\frac{1}{2}\right)^2 &> f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) f\left(\frac{n-2}{n}\right) f\left(\frac{n-1}{n}\right) \\
&\ge \prod_{k=1}^{n-1} f\left(\frac{n}{k}\right) = \prod_{k=1}^{n-1} \frac{4k(n-k)}{n^2} = 4^{n-1} n^{-2(n-1)} \Big(\prod_{k=1}^{n-1} k\Big)\Big(\prod_{k=1}^{n-1} (n-k)\Big) \\
&= 4^{n-1} n^{-2(n-1)} (n-1)! \cdot (n-1)! = 2^{2(n-1)} n^{-2(n-1)} ((n-1)!)^2.
\end{aligned}$$

Taking the square root of this inequality, we obtain

$$\frac{1}{2} > 2^{n-1} n^{-(n-1)} (n-1)!\,,$$

i.e.,

$$\frac{n^{n-1}}{2^n} > (n-1)!\,.$$

Multiplying both sides by n, the inequality to be established follows.

Second solution: The result also can be proved by induction. For $n = 6$ the result is true. Indeed, for $n = 6$ we have

$$\left(\frac{n}{2}\right)^n = 729 \qquad \text{and} \qquad n! = 720.$$

Assuming that this is true for some $n \geq 6$, we have

$$\begin{aligned}\left(\frac{n+1}{2}\right)^{n+1} &= \frac{1}{2}\left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n}{2}\right)^n (n+1) \\ &> \frac{1}{2}\left(\frac{n+1}{n}\right)^n \cdot n!(n+1) = \frac{1}{2}\left(\frac{n+1}{n}\right)^n \cdot (n+1)!,\end{aligned}$$

where the inequality follows by the induction hypothesis. Therefore, the result will follow if we show that

$$\left(\frac{n+1}{n}\right)^n > 2 \tag{1}$$

for $n \geq 6$. This is indeed true, since the left-hand side is equal to 2 for $n = 2$, and the function

$$f(x) \stackrel{def}{=} \left(\frac{x+1}{x}\right)^x > 2$$

is increasing for $x > 0$. Since $\ln x$ is an increasing function, it is enough to show that

$$g(x) \stackrel{def}{=} \ln f(x) = x\bigl(\ln(x+1) - \ln x\bigr)$$

is increasing for $x > 0$.

Assuming $x > 0$, we have

$$g'(x) = \ln(x+1) - \ln x + \frac{x}{x+1} - 1 = \int_x^{x+1} \frac{dt}{t} - \frac{1}{x+1} > 0.$$

The inequality is true since

$$\int_x^{x+1} \frac{dt}{t} > \int_x^{x+1} \frac{dt}{x+1} = \frac{1}{x+1}.$$

This shows that $g(x)$ is increasing; hence so is $f(x)$, and so inequality (1) in fact holds for $n \geq 2$; for $n = 1$, we have equality instead of the inequality. Thus the result follows by induction.

Note. Inequality (1) for integers $n \geq 2$ can also be established by noting that for $n \geq 1$ we have

$$\left(\frac{n+2}{n+1}\right)^{n+1} > \left(\frac{n+1}{n}\right)^n. \tag{2}$$

To see this, observe that by the binomial theorem we have

$$\begin{aligned}\left(\frac{n+1}{n}\right)^n &= \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{n^k} \prod_{j=0}^{k-1} (n-j) \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{k-1} \frac{1}{n} (n-j) = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right).\end{aligned}$$

Replacing n by $n+1$, this equation becomes

$$\left(\frac{n+2}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{n+1}\right).$$

Comparing the right-hand sides of the last two displayed formulas, it is clear that inequality (2) holds. Indeed, for each value of k with $0 \le k \le n$ the the term in the sum on the right-hand side of the latter formula is larger than the corresponding term on the right-hand side of the former. Furthermore, the latter formula has an additional term for $k = n+1$.

5) (SENIOR 5) Assume that $P(x)$ is a polynomial with integer coefficients that assumes the value 7 for four different integer values of x. Show that we cannot have $P(x) = 14$ for any integer x.

Source: Problem 212 on p. 47 in the book D. O. Shklarsky, N. N. Chentzov, I. M. Yaglom, The USSR Olympiad Problem Book, W. H. Freeman and Co., San Francisco and London, 1962.

Solution: We can simplify the problem by replacing $P(x)$ with $P(x) - 7$. We continue to write $P(x)$ for this new polynomial. With this change, the question becomes:

Assume that $P(x)$ is a polynomial with integer coefficients that assumes the value 0 for for different integer values of x. Show that we cannot have $P(x) = 4$ for any integer x.

Under the assumption, we can assume that the polynomial $P(x)$ can be factored as

$$P(x) = (A_1x + B_1)(A_2x + B_2)(A_3x + B_3)(A_4x + B_4)Q(x).$$

According to a well-known theorem of Gauss, we may assume that the coefficients of all the factors on the right-hand side are integers.[1] If we have $P(x_0) = 7$, four of the factors on the right-hand side must be ± 1, and the fifth one must be ± 7 for $x = x_0$; this means that at least three of the first four factors on the right-hand side are equal to ± 1.

Let $A_kx + B_k$ one of these factors. First, the equality $A_kx_0 + B_k = \pm 1$ means that A_k and B_k are relatively prime. Second, since $-B_k/A_k$ is an integer zero of $P(x)$, this means that we must have $A_k = \pm 1$; by multiplying each of these linear factors by ± 1 and compensating for this by multiplying the remaining factors by ± 1, we may assume that $A_k = 1$ for the three factors for which $A_kx_0 + B_k = \pm 1$. Two of these three factors have the same value. That is, two of these factors, say $A_1x_0 + B_1$ and $A_2x_0 + B_2$, are equal. Given that $A_1 = A_2 = 1$, this means that $B_1 = B_2$. That is, these two factors correspond to the same zero of $P(x)$. This contradicts our assumptions.

6) (SENIOR 6) Let b_n for $n \ge 1$ be positive real numbers such that for every sequence of numbers $a_n \ge 0$ such that $a_n \to 0$ the series $\sum_{n=1}^{\infty} a_n b_n$ converges. Prove that then $\sum_{n=1}^{\infty} b_n$ also converges.

Source: Problem 3, The University of British Columbia, Department of Mathematics, Qualifying Examination, Analysis September 5, 2017

`http://www.problemcorner.or://www.math.ubc.ca/Grad/QualifyingExams/`

Solution: Assume, on the contrary, that $\sum_{n=1}^{\infty} b_n$ diverges, and let $s_n = \sum_{k=1}^{n} b_k$. Put $a_n = 1/\sqrt{s_n}$. Then a_n is a nonincreasing sequence such that $a_n \to 0$. Yet we have

$$\sum_{k=1}^{n} a_k b_k \ge \sum_{k=1}^{n} a_n b_k = a_n \sum_{k=1}^{n} b_k = \frac{1}{\sqrt{s_n}} s_n = \sqrt{s_n} \to \infty.$$

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[1] Gauss's theorem says that if a polynomial with integer coefficients can be factors as a product of two polynomials with rational coefficients, then these factors can be replaced by polynomials with integer coefficients; each of these latter polynomials is a constant multiple of the corresponding original factor.

This completes the proof.

7) (SENIOR 7) Writing $\mathbb{Q}$ for the set of rationals, show that there are strictly increasing functions $f, g : \mathbb{Q} \to \mathbb{Q}$, both of them onto $\mathbb{Q}$ such that $f(r) + g(r) \neq 0$ for any rational number r.

Source: Problem A.228, p. 42, Középiskolai Matematikai Lapok, Vol. 50/1, January 2000, Proposed by Ervin Fried. See

`http://db.komal.hu/scan/2000/01/MAT0001.PS.png.41`

Solution: First note that if q_1, q_2, r_1, and r_2 are rational numbers with $q_1 < q_2$ and $r_1 < r_2$ there is a strictly increasing function $\phi : [q_1, q_2) \cap \mathbb{Q} \to [r_1, r_2) \cap \mathbb{Q}$ that is onto. We can simply take ϕ to be the linear function

$$\phi(x) = r_1 + \frac{r_2 - r_1}{q_2 - q_1}(x - q_1).$$

Next note that if α and β are irrational numbers, then there is a strictly increasing function $f : \mathbb{Q} \to \mathbb{Q}$ that is onto such that for all rational x, if $x < \alpha$ then $f(x) < \beta$ and if $x > \alpha$ then $f(x) > \beta$. To define f for rational $x < \alpha$, let α_n and β_n form strictly increasing sequences of rational numbers such that $\alpha_n \to \alpha$ and $\beta_n \to \beta$. Construct f in such a way that f maps $[\alpha_n, \alpha_{n+1}) \cap \mathbb{Q}$ onto $[\beta_n, \alpha_{n+1}) \cap \mathbb{Q}$. As for $x < \alpha_1$, we may assume that $\alpha_1 = \beta_1$, and for $x < \alpha_1$ we can define f as the identity function. For $x > \alpha$, f can be defined in an analogous way, by taking strictly decreasing sequences of rationals.

Finally, to define f and g as required in the problem, take two irrational numbers, α and β, and given any rational x, let f be such that $f(x) < \beta$ if $x < \alpha$, and $f(x) > \beta$ if $x > \alpha$, and let g be such that $g(x) < -\beta$ if $x < \alpha$, and $f(x) > -\beta$ if $x > \alpha$. Then $f(x) + g(x) < 0$ if $x < \alpha$ and $f(x) + g(x) > 0$ if $x > \alpha$. Hence, there is no rational x such that $f(x) + g(x) = 0$.