Senior Prize Exam
Spring 2021

1) Find all positive integers x such that $x^5 - 3x^2 = 216$.

2) If n is an integer, show that

$$\frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$

is also an integer.

3) Let n be a positive integer. Assume we are given n (not necessarily distinct) integers such that their sum is 0 and their product is n itself. Prove that n is divisible by 4.

4) Show that for every integer $n > 5$ we have

$$\left(\frac{n}{2}\right)^n > n!.$$

5) Assume that $P(x)$ is a polynomial with integer coefficients that assumes the value 7 for four different integer values of x. Show that we cannot have $P(x) = 14$ for any integer x.

6) Let b_n for $n \geq 1$ be positive real numbers such that for every sequence of numbers $a_n \geq 0$ such that $a_n \to 0$ the series $\sum_{n=1}^{\infty} a_n b_n$ converges. Prove that then $\sum_{n=1}^{\infty} b_n$ also converges.

7) Writing $\mathbb{Q}$ for the set of rationals, show that there are strictly increasing functions $f, g : \mathbb{Q} \to \mathbb{Q}$, both of them onto $\mathbb{Q}$ such that $f(r) + g(r) \neq 0$ for any rational number r.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2021/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.