All Problems on the Prize Exams
Spring 2022
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Let a and b be integers such that a^2+b^2 is divisible by 7. Show that both a and b are divisible by 7.

Source: Probl 1678, p. 145, Középiskolai Matematikai Lapok, No. 6, Vol. XV, January 1908. See

`http://db.komal.hu/scan/1908/01/90801145.g4.png`

Solution: If one of a and b is divisible by 7, it is easy to see that the other one must also be divisible by 7 for a^2+b^2 to be divisible by 7. So assume that neither a nor b is divisible by 7. For any integer n not divisible by 7, it is clear that $n \equiv \pm 1 \mod 7$, or $n \equiv \pm 2 \mod 7$, or $n \equiv \pm 3 \mod 7$. In these cases we have $n^2 \equiv 1 \mod 7$, or $n^2 \equiv 2 \mod 7$, or $n^2 \equiv 4 \mod 7$, respectively. If we pick any two from among the numbers 1, 2, and 4, allowing to pick the the same number twice (instead of picking different numbers), the picked numbers never add up to 7, it follows that a^2+b^2 is not divisible by 7 if neither a nor b is divisible by 7.

2) (JUNIOR 2 and SENIOR 2) Let a, b, c be integers not all of which are 0, and let $3+\sqrt{5}$ be one of the roots of the equation

$$ax^2+bx+c=0.$$

Show that its other root is $3-\sqrt{5}$.

Source: Probl 2163, p. 184, Középiskolai Matematikai Lapok, No. 8, Vol. XIX, March 1912. See

`http://db.komal.hu/scan/1898/06/89806170.g4.png`

Solution: First note that since $3+\sqrt{5}$ is irrational, we cannot have $a=0$. If $x_1=3+\sqrt{5}$ and x_2 are the solutions of the above equation, then $x_1x_2=c/a$ and $x_1+x_2=-b/a$. Using the first of these equations, we have

$$x_2=\frac{c}{a}\,\frac{1}{x_1}=\frac{c}{a}\,\frac{1}{3+\sqrt{5}}=\frac{4c}{a}\,(3-\sqrt{5}); \tag{1}$$

the last equation holds since

$$(3+\sqrt{5})(3-\sqrt{5})=9-5=4.$$

Thus

$$-\frac{b}{a}=x_1+x_2=(3+\sqrt{5})+\left(\frac{4c}{a}\,(3-\sqrt{5})\right)=3\left(1+\frac{4c}{a}\right)+\sqrt{5}\left(1-\frac{4c}{a}\right)$$

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

Since the left-hand side is rational and everything except for $\sqrt{5}$ on the right-hand side is rational, the coefficient of $\sqrt{5}$ must be 0; that is, $4c/a = 1$. Hence equation (1) shows that we indeed have $x_2 = 3 - \sqrt{5}$.

3) (JUNIOR 3 and SENIOR 3) Show that for every positive integer n we have

$$(2n+1)^n \ge (2n)^n + (2n-1)^n.$$

Source: Problem 462, The problems of the All-Soviet-Union mathematical competitions 1961-1986. See
`http://web.archive.org/web/20120825124642/http://pertselv.tripod.com/RusMath.html`

Solution: Using the binomial theorem, we have

$$\begin{aligned}(2n+1)^n - (2n-1)^n &= \sum_{k=0}^{n} \binom{n}{k} 1^k (2n)^{n-k} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k (2n)^{n-k} \\ &= \sum_{\substack{k=0 \\ k \text{ is odd}}}^{n} 2\binom{n}{k}(2n)^{n-k} \ge 2\binom{n}{1}(2n)^{n-1} = 2n(2n)^{n-1} = (2n)^n;\end{aligned}$$

here the second equation holds since in the sums to its left the terms for even k cancel; for the inequality, we dropped all terms except those corresponding to $k = 1$. This calculation establishes the inequality to be proved.

4) (JUNIOR 4) Let a, b, c be integers such that

$$a^2 + c^2 = 2b^2.$$

Show that $c^2 - a^2$ is divisible by 48.

Source: Probl 533, p. 170, Középiskolai Matematikai Lapok, No. 10, Vol. V, June 1898. See
`http://db.komal.hu/scan/1898/06/89806170.g4.png`

Solution: We need to show that $c^2 - a^2$ is divisible by 3 and 16. As for divisibility by 3, this is clear if both a and c are divisible by 3, so assume that this is not the case.

Next, note that the equation cannot be satisfied if exactly one of a and c is divisible by 3. For example, if a is not divisible by 3, then $a \equiv \pm 1 \mod 3$, so $a^2 \equiv 1 \mod 3$. If now c is divisible by 3, then $2b^2 \equiv 1 \mod 3$, and so $b^2 \equiv 2 \equiv -1 \mod 3$, which is impossible. The case when a is divisible by 3 and c is not is similar.

Now assume that neither a nor c are divisible by 3. Then $a^2 \equiv c^2 \equiv 1 \mod 3$, so $c^2 - a^2$ is divisible by 3.

To show that $c^2 - a^2$ is divisible by 16, first note that for the equation above to be satisfied, $a^2 + c^2$ must be even, so a and c must have the same parity. If both a and c are even, then $a^2 + c^2$ is divisible by 4, so b is also even according to the equation above. In this case, there are integers k, l, and m such that $a = 2k$, $b = 2l$, and $c = 2m$. Then the above equation can be written as

$$(2k)^2 + (2m)^2 = 2(2l)^2,$$

that is,

$$k^2 + m^2 = 2l^2,$$

This means that k, l, and m satisfy an equation analogous to the one satisfied by a, b, and c. Then, according to the assertion of the theorem, yet to be proved, $m^2 - k^2$ must be divisible by 16, from which it follows that $c^2 - a^2 = 4(m^2 - k^2)$ is also divisible by 16. That is, we can keep dividing a, b, and c by 2 until the numbers replacing a and c are both odd; since the numbers so obtained satisfy an equation similar to the one above, this amounts to saying that in the above equation both a and c are odd. Making this assumption, we have $a^2 \equiv c^2 \equiv 1 \mod 4$, and so $2b^2 \equiv 2 \mod 4$. This cannot be true if b is even; hence we may assume that all of a, b, and c are odd. According to the above equation, we have

$$c^2 - a^2 = 2c^2 - (a^2 + c^2) = 2c^2 - 2b^2 = 2(c + b)(c - b).$$

So, we need to prove that if both c and b are odd, then $(c+b)(c-b)$ is divisible by 8. We distinguish two cases.

In the first case, assume that both c and b are congruent to 1, or both are congruent to -1 modulo 4. In this case $c - b$ is divisible by 4 and $c + b$ is divisible by 2; so $(c + b)(c - b)$ is indeed divisible by 8.

In the second case, assume that one of c and b is congruent to 1 modulo 4 and the other one is congruent to -1 modulo 4. In this case, $c + b$ is divisible by 4 and $c - b$ is divisible by 2. So, again, $(c + b)(c - b)$ is divisible by 8.

5) (JUNIOR 5) Find all (complex) solutions of the equation

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

The solutions must be expressed in terms of the four operations and roots (such as square roots, cube roots, fourth roots, etc.).

Solution: First note that

$$(z - 1)(z^4 + z^3 + z^2 + z + 1) = z^5 - 1, \tag{1}$$

so all solutions of the equation in the problem satisfy the the equation $z^5 - 1 = 0$; that is, for any z satisfying the former equation we have $z^5 = 1$. Hence $z^4 = z^{-1}$ and $z^2 = z^{-2}$. Thus the equation can be written as

$$z + z^{-1} + z^2 + z^{-2} + 1 = 0. \tag{2}$$

Writing

$$w = z + z^{-1}, \tag{3}$$

we have $w^2 = z^2 + 2 + z^{-2}$; hence equation (2) can be written as

$$w^2 + w - 1 = 0.$$

The quadratic formula gives

$$w = \frac{-1 \pm \sqrt{5}}{2}.$$

Hence, by equation (3) we have

$$z^2 - \frac{-1 \pm \sqrt{5}}{2} z + 1 = 0. \tag{4}$$

That is, taking the + sign in the $\pm$ in (4), we have

$$z_{1,4} = \frac{\frac{-1+\sqrt{5}}{2} \pm \sqrt{\left(\frac{-1+\sqrt{5}}{2}\right)^2 - 4}}{2} = \frac{-1 + \sqrt{5} \pm \sqrt{-10 - 2\sqrt{5}}}{4} = \frac{-1 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}}{4},$$

where z_1 refers to the case when the + sign is taken before the imaginary part, and z_4, when the $-$ sign is taken; the numbering of z_k will be explained in a note below. Similarly, taking the $-$ sign in the $\pm$ in (4), we have

$$z_{2,3} = \frac{\frac{-1-\sqrt{5}}{2} \pm \sqrt{\left(\frac{-1-\sqrt{5}}{2}\right)^2 - 4}}{2} = \frac{-1 - \sqrt{5} \pm \sqrt{-10 + 2\sqrt{5}}}{4} = \frac{-1 - \sqrt{5} \pm i\sqrt{10 - 2\sqrt{5}}}{4},$$

where z_2 refers to the case when the + sign is taken before the imaginary part, and z_3, when the $-$ sign is taken. This gives all four solutions of the equation.

Note: For a complex number ζ, its absolute value $|\zeta|$ is its distance from the origin 0 when graphed in the complex plane; its argument $\arg \zeta$ is the angle described by the rotation of the vector 01 to the ray 0ζ, with counterclockwise rotation counting as positive; the argument is only determined up to an integer multiple of 2π. For two complex numbers ζ_1 and $\zeta 2$ we have

$$|\zeta_1 \zeta_2| = |\zeta_1||\zeta_2| \qquad \text{and} \qquad \arg(\zeta_1 \zeta_2) = \arg \zeta_1 \arg \zeta_2. \tag{5}$$

In view of (1), the equation $z^5 = 1$ has the solutions $z_5 = 1$ in addition to the solutions z_1, z_2, z_3, z_4 listed above. In view of (5), it is clear that $|z_k| = 1$ for $k = 1, 2, 3, 4, 5$, that is, each z_k lies on the unit circle. It is also clear from (5) that the arguments of these z_k are $2\pi l/5$ for $l = 1, 2, 3, 4, 5$.[1] In fact, it is easy to ascertain that the numbering of z_k so chosen that $\arg z_k = 2\pi k/5$ for $l = 1, 2, 3, 4, 5$. Thus, the numbers z_k are the vertices of a regular pentagon inscribed in the unit circle; the distance of any two of these vertices is the side of the regular pentagon inscribed into a circle of radius 1.

The vertices z_2 and z_3 are adjacent and lie on the same vertical line; so their distance is

$$\frac{1}{i}(z_2 - z_3) = \frac{\sqrt{10 - 2\sqrt{5}}}{2} = \sqrt{\frac{5 - \sqrt{5}}{2}}.$$

This is the length of the side of the regular pentagon inscribed in the unit circle.

Using this result, it is easy to describe how to construct a regular pentagon; the following description was given by Claudius Ptolemy (circa 100–circa 170 AD):

Let AB a diameter of a circle with center O and let D be a point on the circle such that $OD \perp AB$ Let C be the midpoint of the line segment OB, and let E be the point on the line segment AO such that $CE = CD$. Then the length of the line segment DE is the side of the regular pentagon

[1] For $l = 5$ this gives the argument 2π which is the argument of $z_5 = 1$; this argument can also be taken 0, since the argument is only determined up to a multiple of 2π.

inscribed in the given circle. Using the Pythagorean theorem twice, it is easy to verify that this construction is correct.

6) (JUNIOR 6) Let m be a positive integer. Given $2m+1$ different integers, each of absolute value not greater than $2m-1$, show that it is possible to choose three numbers among them such that their sum is 0.

Source: Problem 367, The problems of the All-Soviet-Union mathematical competitions 1961-1986. See
`http://web.archive.org/web/20120825124642/http://pertselv.tripod.com/RusMath.html`

Solution: Let S be a set of integers with absolute value $\leq 2m-1$ such that the sum of any three of its elements is nonzero. We will show that the number of elements of S is $\leq 2m$. Before we do this, note that the set of all odd integers in the given range has the desired property, and it has exactly $2m$ elements, so the bound given is exact.

Let s_0 be the element of S that has the smallest absolute value; note that if $s_0 \neq 0$, $-s_0$ may also be an element of S, so the choice of s_0 is not necessarily unique. In any case, without loss of generality we may assume that $s_0 \geq 0$. Assume S has k elements in the range $(s_0, 2m-1-s_0]$ (this is interpreted as the empty set if $s_0 \geq 2m-1-s_0$). This disallows k elements of S in the range $[-2m+1, -s_0)$, since $-(s_0+s]$ for $s \in S \cap (s_0, 2m-1-s_0]$ cannot be an element of S, because no three elements of S may sum to 0. This disallows k elements of S in the range $(0, -2m-1]$. A further s_0 elements of the range $(-2m+1, 0]$ are excluded from among the elements s of S with $s < s_0$, since none of these elements can be in the range $(-s_0, 0]$. Indeed, 0 cannot be selected as s in case $s_0 = 0$ since we must have $s < s_0$. It cannot be selected in case $s_0 > 0$ either, since we have $|s| \geq |s_0|$. The elements other than 0 in the range $(-s_0, 0]$ cannot be selected, either, since $|s| \geq |s_0|$. Since the range $[-2m+1, 0)$ has $2m$ elements, this allows $2m-1-k-s_0$ elements s with $s < s_0$ to be in S.

To count the number of elements of S in the range $[s_0, 2m-1)$, among these, we can count s_0 plus the k elements in the range $(s_0, 2m-1-s_0]$ Even allowing that all numbers in the range $(2m-1-s_0, 2m-1]$ are elements of S, this adds a further $(2m-1)-(2m-1-s_0) = s_0$ elements s to S with $s \geq s_0$. This means that the total number of elements s of S with $s \geq s_0$ is at most $1+k+s_0$. Since the maximum number of elements s of S with $s < s_0$ is $2m-1-k-s_0$, this means that S has at most $(1+k+s_0)+(2m-1-k-s_0) = 2m$ elements. This completes the proof.

7) (JUNIOR 7) Let G be the set of all permutations of the set $\mathbb{Z}$ of all integers (positive, negative, or zero); that is, the elements of G are one-to-one mappings of $\mathbb{Z}$ onto itself. For $f, g \in G$ write $fg \stackrel{def}{=} f \circ g$ (i.e., f composition g). Write id for the identity permutation (i.e., $\mathrm{id}(k) = k$ for all $k \in \mathbb{Z}$). For a positive integer n, write $f^n \stackrel{def}{=} f \circ f \circ \ldots \circ f$ (f repeated n times).

Give an example of $f, g \in G$ such that $f^2 = g^2 = \mathrm{id}$ and yet $(fg)^n \neq \mathrm{id}$ for any positive integer n.

Source: Based on: Algebra Qualifying Exam, May 2017, p. 2, Problem 1d, Qualifying Exams in Mathematics, University of Missouri. The original problem is formulated in a group theory language; we avoided this, so as to make the problem more widely accessible. See

`http://www.problemcorner.or://www.math.missouri.edu/grad/qualifying-exams`

Solution: For an arbitrary $k \in \mathbb{Z}$, put

$$f(2k) = 2k-1 \quad \text{and} \quad f(2k-1) = 2k$$

and

$$g(2k) = 2k+1 \quad \text{and} \quad g(2k+1) = 2k$$

It is then easy to see that $f^2 = g^2 = \mathrm{id}$. Further, for each $k \in \mathbb{Z}$ we have $fg(2k) = 2k+2$. Hence we do not have $(fg)^n = \mathrm{id}$ for any positive integer n; indeed, we have $(fg)^n(0) = 2n$.

8) (SENIOR 4) Let $f : [0,1] \to [0,1]$ be a continuous function. Show that $f(x) = x$ for some $x \in [0,1]$. (You can use without proof any result described in basic calculus courses or any result proved in standard advanced calculus courses.)

Source: Basic Exam in Advanced Calculus/Linear Algebra, Fall 2003, University of Massachusetts Amherst. See

`https://www.math.umass.edu/graduate/sample-qualifying-exams`

Solution: Write $g(x) = f(x) - x$. Then g is continuous on $[0,1]$, $g(0) \ge 0$, and $g(1) \le 0$. Hence $g(x) = 0$ for some $x \in [0,1]$ by the Intermediate Value Theorem. For this x we have $f(x) = x$.

9) (SENIOR 5) Let f be a differentiable function on the real line such that $f(0) = 0$ and $f'(x) > f(x)$ for all real x. Show that $f(x) > 0$ for all $x > 0$.

Source: Problem 6.20, p. 101 in Asuman G. Aksoy and Mohamed A. Khamsi, A Problem Book in Real Analysis, Springer, New York, Dordrecht, Heidelberg, London, 2009.

Solution: According to the assumption, we have $f'(0) > f(0) = 0$; that is,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} > 0;$$

therefore, there is an $\epsilon > 0$ such that

$$f(x) > 0 \qquad \text{for all } x \text{ with } 0 < x < \epsilon. \tag{1}$$

Assume that $f(x_0) < 0$ for some $x_0 > 0$. Then there is an x with $0 < x < x_0$ such that $f(x) = 0$ by the Intermediate-Value Theorem, since f, being differentiable, is continuous. Let

$$x_1 = \min\{x > 0 : f(x) = 0\}.$$

This minimum exists, since the set on the right-hand side is closed in view of (1), given that f is continuous. Clearly we have $x_1 > 0$ and $f(x_1) = 0$. By the Mean-Value Theorem of Differentiation, there is a ξ with $0 < \xi < x_1$ such that

$$f'(\xi)x_1 = f'(x_1)(x_1 - 0) = f(x_1) - f(0) = 0.$$

Hence $f(\xi) < f'(\xi) = 0$ in view of our assumptions, which is a contradiction, since $f(\xi) > 0$ for all ξ with $0 < \xi < x_1$ by the definition of x_1. This contradiction shows that $f(x) > 0$ for all $x > 0$.

10) (SENIOR 6) Let $n \ge 3$ be an odd integer, and for k with $0 \le k \le n$ let p_k be a polynomial of degree k. Assume, further, that $p'_k(x) = p_{k-1}(x)$ for k with $1 \le k \le n$, where the prime indicates derivative. Finally, assume that $p_k(-1) = p_k(0) = p_k(1)$ for odd k with $3 \le k \le n$. Show that $p_n(x) \ne 0$ for $x \ne 0$ with $-1 < x < 1$.

Solution: Assume, on the contrary that there is an x_n with $x_n \ne 0$ and $-1 < x_n < 1$ such that $p_n(x_n) = 0$. We will then show by induction on k stepping down from $k = n$ through odd numbers to $k = 3$ that that given any odd k with $3 \le k \le n$, there is an x_k with $x_k \ne 0$ and $-1 < x_k < 1$ such that $p_k(x) \ne 0$. For $k = 3$ this, of course represents a contradiction, since it means that the degree 3 polynomial p_3 has four distinct zeros: x_3, -1, 0, and 1.

Indeed, assume, there is an x_k with $x_k \neq 0$ and $-1 < x_k < 1$ such that $p_k(x) \neq 0$. for some odd k with $3 < k \leq n$. We will then show that the analogous statement is also true with $k-2$ replacing k. Indeed, noting that $p_{k-1} = p'_k$ by Rolle's theorem, there must be pairwise distinct real numbers ξ_1, ξ_2, and ξ_3 such that $p_{k-1}(x_1) = p_{k-1}(x_1) = p_{k-1}(x_1) = 0$; ξ_1, ξ_2, and ξ_3 are located in the three open intervals determined by the adjacent ones among the numbers -1, 0, 1, and x_k. We may assume that $\xi_1 < \xi_2 < \xi_3$. Then, noting that $p_{k-2} = p'_{k-1}$ using Rolle's theorem again, there are η_1 and η_2 with $\xi_1 < \eta_1 < \xi_2 < \eta_2 < \xi_3$. such that $p_{k-2}(\eta_1) = p_(\eta_2) = 0$. We have $-1 < \eta_1 < \eta_2 < 1$, so we have $\eta_1 \neq 0$ or $\eta_2 \neq 0$. Pick $x_{k-2} = \eta_1$ or $x_{k-2} = \eta_2$ such that $x_{k-2} \neq 0$. This shows that the above statement is true with $k-2$ replacing k, completing the induction and establishing the assertion of for all odd k with $3 \leq k \leq n$.

As we remarked above, for $k = 3$ this is a contradiction, completing the proof of the assertion in the problem.

Note: The problem is based on a property of the Bernoulli polynomials $B_k(x)$. These polynomials are defined by the equation

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k,$$

where, for fixed x, the series on the right is the Taylor expansion at $z = 0$ of the function on the left-hand side. It can be show $B_k(x)$ is a polynomial of degree k of x, and $B'_{k+1}(x) = B_k(x)$ for all $k \geq 0$. Further $B_k(1/2) = 0$ for odd $k \geq 1$, and $B_k(0) = B_k(1) = 0$ for odd $k \geq 3$. Using these properties, one can show by the argument used in the solution of the above problem that $B_k(x)$ cannot have zeros other than $x = 1/2$ in the interval $(0, 1)$, and the zero at $x = 1/2$ is simple (i.e., it has multiplicity 1). This fact is important for obtaining a useful form of the remainder term of the Euler–Maclaurin summation formula. See Section 24, The Euler–Maclaurin Summation Formula, pp. 84–93, in my notes Introduction to Numerical Analysis with C Programs at

http://www.sci.brooklyn.cuny.edu/˜mate/nml/

as the file `numanal.pdf`. According to what we said about the Bernoulli polynomials, the polynomials

$$p_k(x) = \alpha_k B_k\big((x+1)/2\big) \tag{1}$$

with appropriate constants α_k satisfy the assumptions of the problem.

It is also easy to see that the polynomials described in (1) are the only polynomials satisfying the assumptions, since the polynomials are uniquely determined, aside from a multiplicative constant. This is because $p_3(x) = c(x-1)x(x+1)$ for some constant $c \neq 0$, and if $p_k(x)$ is uniquely determined for some odd integer $k \geq 3$, then p_{k+2} is also uniquely determined by the requirement that $p''_{k+2}(x) = p_k(x)$ and $p_{k+2}(1) = p_{k+2}(0) = 0$. Indeed, the coefficient of the term x^l for $l \geq 2$ in polynomial $p_{k+2}(x)$ is determined by the equation $p''_{k+2}(x) = p_k(x)$. The requirement that $p_{k+2}(0) = 0$ implies that the constant term of $p_{k+2}(x)$ is 0, and then the requirement that $p_{k+2}(1) = 0$ allows one to determine the coefficient of x in $p_{k+2}(x)$.

Here we did not make use of the equation $p_{k+2}(-1) = 0$. This equation may in fact cause trouble in that it may give a different value for the coefficient of x in $p_{k+2}(x)$ that we obtained from the equation $p_{k+2}(1) = 0$. That this does not happens would need to be proven, but since the Bernoulli polynomials show that the polynomials p_k as described do exist, this we do not need to prove this.

11) (SENIOR 7) Given integers m and n with $0 \le m \le n$, show that

$$\sum_{k=0}^{n} \frac{(-1)^k k^m}{k!(n-k)!} = \begin{cases} (-1)^n & \text{if } m = n, \\ 0 & \text{if } 0 \le m < n. \end{cases}.$$

Here, in case $k = 0$ and $m = 0$, take $k^m = 1$ (normally, 0^0 is undefined). Further, given a nonnegative integer l, $l! \overset{def}{=} \prod_{k=1}^{l} k$; that is, for $l = 0$ we have $l! = 1$, and for $l \ge 1$ we have $l! = 1 \cdot 2 \cdot \ldots \cdot l$.

Source: Based on Problem 3/1, May 2018, PhD Entrance Examination, p. 7, Institute of Mathematics, Faculty of Science, Eötvös Loránd University, Budapest, Hungary (Eötvös Loránd uses the Hungarian name order; in English speaking countries, he is known as Roland von Eötvös, the inventor of the Eötvös pendulum, a variation of the torsion balance). See

`https://www.math.elte.hu/en/phd-entrance-exam/`

Solution: Throughout the discussion below, we use the interpretation $0^0 = 1$ wherever it occurs as a result of substitution certain values for the variables in an expression.[2] The identity to be proved is certainly true in case $n = m = 0$, since the sum on the left-hand side only has a single term, and this term equals 1. It is also true in case $n > m = 0$. Indeed, in this case, the sum on the left-hand side is

$$\text{(1)} \qquad \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \frac{1}{n!}(1-1)^n = 0;$$

here the first equation is true since

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and the second one holds by the Binomial Theorem.[3]

We have yet to establish the identity in case $n \ge m \ge 1$. Using induction, assume that above identity is true with $n' = n - 1$ and m' with $0 \le m' \le n'$ replacing n and m. In view of (1), we have

$$\begin{aligned} -\sum_{k=0}^{n} \frac{(-1)^k k^m}{k!(n-k)!} &= \sum_{k=0}^{n} \frac{(-1)^k (n^m - k^m)}{k!(n-k)!} = \sum_{k=0}^{n-1} \frac{(-1)^k (n^m - k^m)}{k!(n-k)!} \\ \text{(2)} \qquad &= \sum_{k=0}^{n-1} \frac{(-1)^k (n-k)}{k!(n-k)!} \sum_{l=0}^{m-1} k^l n^{m-1-l} = \sum_{l=0}^{m-1} n^{m-1-l} \sum_{k=0}^{n-1} \frac{(-1)^k k^l (n-k)}{k!(n-k)!} \\ &= \sum_{l=0}^{m-1} n^{m-1-l} \sum_{k=0}^{n-1} \frac{(-1)^k k^l}{k!(n-1-k)!}; \end{aligned}$$

here the second equation is true since the summand to its left is 0 in case $k = n$. The third equation follows from the identity[4]

$$n^m - k^m = (n-k) \sum_{l=0}^{m-1} k^l n^{m-1-l}.$$

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[2]Note that the $0^0 = 1$ convention is common when writing polynomials or power series using the $\sum$ notation.
[3]Note that here we used the stipulation that $0^0 = 1$.
[4]In this identity, we also need to take the interpretation $0^0 = 1$ when $k = 0$.

Finally, the last equation‘holds since $(n-k)! = (n-k)(n-1-k)!$.

By the induction hypothesis, the inner sum on the right-hand side of (2) is 0 unless $l = n-1$. If $l = n-1$, which is possible only in case $m = n$, this sum is $(-1)^{n-1}$. Since the left-hand side equals the left-hand side of the identity to be established, this identity is indeed valid.

Note: The problem source states the identity to be proved as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n = n!. \tag{3}$$

Noting that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

and dividing both sides of this identity by $n!$, we obtain

$$\sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!} (n-k)^n = 1.$$

Using l with $l = n-k$ for the summation variable instead of k, we obtain

$$\sum_{l=0}^{n} (-1)^{n-l} \frac{1}{l!(n-l)!} l^n = 1.$$

Multiplying both sides by $(-1)^n$, noting that $(-1)^{-l} = (-1)^l$, and writing k instead of l for the summation valuable, we obtain

$$\sum_{k=0}^{n} \frac{(-1)^k k^m}{k!(n-k)!} = (-1)^n.$$

This is the case $m = n$ of the identity stated in the problem. We extended the statement to allow $0 \le m \le n$; this is needed in order to support the induction in the proof; this makes the solution of the problem easier.

The problem appeared in the Combinatorics section of the exam. We wonder whether the reason for this was that the identity in question perhaps has a direct combinatorial interpretation. We did not examine this question.

Induction proof of (3)**:** In order to prove (3) by induction, we extend it in a similar way as we stated the problem above. Given integers m and n with $0 \le m \le n$, we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = \begin{cases} n! & \text{if } m = n, \\ 0 & \text{if } 0 \le m < n. \end{cases}; \tag{4}$$

similarly, as in the statement of the problem above, we need to take $(n-k)^m = 1$ in case $n = k$ and $m = 0$. With this stipulation, equation (4) is true in case $m = n = 0$; it is also true for all $n \ge 1$ if $m = 0$ in view of (1).

To establish the identity in case $n \ge m \ge 1$, assume that it is true with $n' = n-1$ and m' with $0 \le m' \le n'$ replacing n and m. Extending the definition of the binomial coefficient with the stipulation that

$$\binom{n}{k} = 0 \qquad \text{if} \qquad k < 0 \text{ or } k > n$$

we can drop the limits of the summation in (4) and take the sum from $-\infty$ to ∞; further, the identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

is valid for all integers $n \geq 1$ and for all k. Thus, the left-hand side of (4) becomes

$$\begin{aligned}\sum_{k=-\infty}^{\infty} (-1)^k \binom{n}{k} (n-k)^m &= \sum_{k=-\infty}^{\infty} (-1)^k \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) (n-k)^m \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \binom{n-1}{k} \big((n-k)^m - (n-k-1)^m \big).\end{aligned} \tag{5}$$

The last equation is the result of what is called Abel rearrangement or partial summation, as we are to explain.

Partial summation (or summation by parts) is described by the formula

$$\sum_{k=\mu}^{\nu} a_k (b_k - b_{k-1}) = a_\mu b_{\mu-1} - a_{\nu+1} b_\nu + \sum_{k=\mu}^{\nu} (a_k - a_{k+1}) b_k$$

for integers μ and ν with $\mu \leq \nu$ and numbers a_k, b_k. This is easy to verify, since each term on one side can be matched up with a corresponding term on the other side. If a_k and b_k are two-way infinite sequences with at least one of them having only finitely many nonzero terms (so that there are no issues of convergence), the identity can be more simply written as

$$\sum_{k=-\infty}^{\infty} a_k (b_k - b_{k-1}) = \sum_{k=-\infty}^{\infty} (a_k - a_{k+1}) b_k.$$

Above, this identity was used with

$$a_k = (n-k)^m \qquad \text{and} \qquad b^k = (-1)^k \binom{n-1}{k}.$$

Continuing the calculation above by using the Binomial Theorem for $(n-k-1)^m = \big((n-k) + (-1)\big)^m$, the right-hand side of (5) equals

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} (-1)^k \binom{n-1}{k} \big((n-k)^m - \big((n-k)-1\big)^m \big) \\ &\quad = \sum_{k=-\infty}^{\infty} (-1)^k \binom{n-1}{k} \left((n-k)^m - \sum_{l=0}^{m} \binom{m}{l} (n-k)^{m-l} (-1)^l \right) \\ &\quad = -\sum_{k=-\infty}^{\infty} (-1)^k \binom{n-1}{k} \sum_{l=1}^{m} \binom{m}{l} (n-k)^{m-l} (-1)^l \\ &\quad = -\sum_{l=1}^{m} (-1)^l \binom{m}{l} \sum_{k=-\infty}^{\infty} (-1)^k \binom{n-1}{k} (n-k)^{m-l}.\end{aligned}$$

By the induction hypothesis, all terms of the inner sum are 0 except in case $m = n$ and $l = 1$, in which case it is $(n-1)!$. This shows that the left-hand side of (5) (which is the same as the left-hand side of (4)) is 0, except in case $m = n$, in which case it is

$$\binom{n}{1}(n-1)! = n(n-1)! = n!.$$

This completes the proof of (4).

A simpler proof involving derivatives: Given a real x and integers m and n with $n \ge m \ge 0$, we will show that

$$f(x,n,m) \overset{def}{=} \sum_{k=0}^{n} \frac{(-1)^k(k+x)^m}{k!(n-k)!} = \begin{cases} (-1)^n & \text{if } m = n, \\ 0 & \text{if } 0 \le m < n. \end{cases} \tag{6}$$

This of course immediately follows from the statement in the problem, since if we expand $(k+x)^m$ with the aid of the binomial theorem, the coefficients of all powers of x are 0 except for that of x^0, according to the equation stated in the problem. It is, however, easier to prove equation (6) directly by induction, and then to establish the statement in the problem above as a consequence. Indeed, it is easy to see that $f(x,0,0) = 1$, and that we have $f(x,n,0) = 0$ according to (1) if $n > 0$. Using induction, given integers $m > 0$, $n \ge m$, and real x, assume that (6) is valid if m, n, and x are replaced by m', n', and x', where $m' \le m$, $n' \le n$, with strict inequality holding in at least one of these inequalities, given that $0 \le m' \le n'$ are integers and x' is a real.

For a start, it immediately follows that $f(x,n,m)$ does not depend on x, since $(d/dx)f(x,n,m) = mf(x,n,m-1) = 0$ by the induction hypotheses. We have

$$\begin{aligned} f(x,n,m) = f(-n,n,m) &= \sum_{k=0}^{n} \frac{(-1)^k(k-n)^m}{k!(n-k)!} = \sum_{k=0}^{n-1} \frac{(-1)^k(k-n)^m}{k!(n-k)!} \\ &= -\sum_{k=0}^{n-1} \frac{(-1)^k(k-n)^{m-1}}{k!(n-k-1)!} = -f(-n,n-1,m-1); \end{aligned}$$

here the first equality holds, since, as we pointed out, $f(x,n,m)$, does not depend on x. The second one holds since the term corresponding to $k = n$ to its left is 0. The third one holds since $(n-k)! = (n-k)(n-k-1)!$. Finally, the right-hand side equals $(-1)^m$ if $n = m$ and it is 0 if $m < n$, by the induction hypothesis. This completes the proof.