JUNIOR PRIZE EXAM
SPRING 2022

1) Let a and b be integers such that a^2+b^2 is divisible by 7. Show that both a and b are divisible by 7.

2) Let a, b, c be integers not all of which are 0, and let $3+\sqrt{5}$ be one of the roots of the equation

$$ax^2+bx+c=0.$$

Show that its other root is $3-\sqrt{5}$.

3) Show that for every positive integer n we have

$$(2n+1)^n \geq (2n)^n+(2n-1)^n.$$

4) Let a, b, c be integers such that

$$a^2+c^2=2b^2.$$

Show that c^2-a^2 is divisible by 48.

5) Find all (complex) solutions of the equation

$$z^4+z^3+z^2+z+1=0.$$

The solutions must be expressed in terms of the four operations and roots (such as square roots, cube roots, fourth roots, etc.).

6) Let m be a positive integer. Given $2m+1$ different integers, each of absolute value not greater than $2m-1$, show that it is possible to choose three numbers among them such that their sum is 0.

7) Let G be the set of all permutations of the set $\mathbb{Z}$ of all integers (positive, negative, or zero); that is, the elements of G are one-to-one mappings of $\mathbb{Z}$ onto itself. For $f, g \in G$ write $fg \overset{def}{=} f \circ g$ (i.e., f composition g). Write id for the identity permutation (i.e., $\mathrm{id}(k)=k$ for all $k \in \mathbb{Z}$). For a positive integer n, write $f^n \overset{def}{=} f \circ f \circ \ldots \circ f$ (f repeated n times).

Give an example of $f, g \in G$ such that $f^2=g^2=\mathrm{id}$ and yet $(fg)^n \neq \mathrm{id}$ for any positive integer n.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2022/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.