Senior Prize Exam
Spring 2022

1) Let a and b be integers such that a^2+b^2 is divisible by 7. Show that both a and b are divisible by 7.

2) Let a, b, c be integers not all of which are 0, and let $3+\sqrt{5}$ be one of the roots of the equation

$$ax^2+bx+c=0.$$

Show that its other root is $3-\sqrt{5}$.

3) Show that for every positive integer n we have

$$(2n+1)^n \ge (2n)^n + (2n-1)^n.$$

4) Let $f:[0,1]\to[0,1]$ be a continuous function. Show that $f(x)=x$ for some $x\in[0,1]$. (You can use without proof any result described in basic calculus courses or any result proved in standard advanced calculus courses.)

5) Let f be a differentiable function on the real line such that $f(0)=0$ and $f'(x)>f(x)$ for all real x. Show that $f(x)>0$ for all $x>0$.

6) Let $n\ge 3$ be an odd integer, and for k with $0\le k\le n$ let p_k be a polynomial of degree k. Assume, further, that $p_k'(x)=p_{k-1}(x)$ for k with $1\le k\le n$, where the prime indicates derivative. Finally, assume that $p_k(-1)=p_k(0)=p_k(1)$ for odd k with $3\le k\le n$. Show that $p_n(x)\ne 0$ for $x\ne 0$ with $-1<x<1$.

7) Given integers m and n with $0\le m\le n$, show that

$$\sum_{k=0}^{n}\frac{(-1)^k k^m}{k!(n-k)!}=\begin{cases}(-1)^n & \text{if } m=n,\\ 0 & \text{if } 0\le m<n.\end{cases}.$$

Here, in case $k=0$ and $m=0$, take $k^m=1$ (normally, 0^0 is undefined). Further, given a nonnegative integer l, $l!\overset{def}{=}\prod_{k=1}^{l}k$; that is, for $l=0$ we have $l!=1$, and for $l\ge 1$ we have $l!=1\cdot 2\cdot\ldots\cdot l$.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2022/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.