All Problems on the Prize Exams
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. The Senior Prize Exam was not given this year.

1) (JUNIOR 1) Show that if the roots of the equation

$$x^2 + px + q = 0$$

with real coefficients are real, then the roots of the equation

$$x^2 + px + q + (x + a)(2x + p) = 0$$

are also real for all real values of a.

Source: Problem 880, p. 110, Középiskolai Matematikai Lapok, No. 4–5, Vol. VIII, December 1900. See

`http://db.komal.hu/scan/1900/12/90012110.g4.png`

Solution: In both equations, the coefficient of x^2 is positive; hence both polynomials are positive if x is large enough. The assumption that the first equation has real roots is equivalent to saying that its minimum is ≤ 0. This minimum is assumed at $x = -p/2$; that is, the value of the first polynomial is ≤ 0 at $x = -p/2$. The second polynomial assumes the same value at this value of x, since the term added is 0 at $x = -p/2$. That is, the second polynomial is also ≤ 0 at this points; thus, the second polynomial also has real roots.

Note: The problem is somewhat tricky, in the sense that the natural instinct when trying to solve the problem would tempt one to look at the discriminant of the equation. This, however, would lead to unnecessary complications, as seen from the above solution.

2) (JUNIOR 2) Find all prime numbers p such that $p^2 + 2$ is also a prime.

Source: Problem 122, p. 60, proposed by Tibor Szele, Középiskolai Matematikai Lapok, No. 3, Vol. I, December 1947. See

`http://db.komal.hu/scan/1947/12/94712060.g4.png`

Solution: Trying $p = 3$, we find that $p = 3$ is such a prime, since $3^2 + 2 = 11$ is also a prime. For all the other primes p, 3 is not a divisor of p; that is, we have $p = 3k \pm 1$ for some integer k. Then $p^2 + 2 = (9k^2 \pm 6k + 1) + 2 = 3(3k^2 \pm 2k + 1)$ is divisible by 3. That is, the only prime satisfying the requirements is $p = 3$.

3) (JUNIOR 3) Let $n \ge 1$ be an odd integer, and let $a_1, a_2, \dots, a_n$ be an arbitrary rearrangement of the numbers $1, 2, \dots, n$. Show that the product

$$(a_1 - 1)(a_2 - 2) \dots (a_n - n)$$

is even.

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

Source: Problem 3, 13th Eötvös Student Competition, Hungary, 1906. See
`https://matek.fazekas.hu/index.php?option=com_content&view=article&id=316:kurschak-jozsef-matematikai-tanuloverseny&catid=26&Itemid=185`

Solution: Write $S = \{1, 2, \dots, n\}$, and for $k \in S$, write $\sigma(k) = a_k$; that is, σ is a permutation of the set S (i.e., a one-to-one mapping of S onto itself). Since n is odd, S contains one more odd numbers than even numbers; so, there must be an odd k such that $\sigma(k) = a_k$ is also odd. Then $a_k - k$ is even; hence the product in question has at least one even factor. Thus, the product is indeed even.

4) (JUNIOR 4) Let x, y, and z be complex numbers such that

$$x + y + z = 0$$

and

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

Show that then

$$x^3 = y^3 = z^3.$$

Source: Based on Problem 880, p. 110, Középiskolai Matematikai Lapok, No. 4–5, Vol. VIII, December 1900. See
`http://db.komal.hu/scan/1900/12/90012110.g4.png`

Solution: By the first equation we have

$$z = -x - y; \tag{1}$$

by the second equation,

$$\frac{1}{z} = -\frac{1}{x} - \frac{1}{y},$$

that is, taking reciprocals,

$$z = -\frac{xy}{x + y}.$$

Multiplying this equation by equation (1), we obtain

$$z^2 = xy. \tag{2}$$

Multiplying both sides by z, we obtain

$$z^3 = xyz.$$

By permuting the variables x, y, and z, in a similar way we also obtain that $x^3 = xyz$ and $y^3 = xyz$. Thus, $x^3 = y^3 = z^3$ follows.

Note: In the original statement of the problem, under the same assumptions, one is asked to show that

$$\frac{x^6 + y^6 + z^6}{x^3 + y^3 + z^3} = xyz.$$

This conclusion easily follows from the assertion $x^3 = y^3 = z^3 = xyz$ we established.

Equations (1) and (2) show that x and y are solutions in t of the quadratic equation

$$t^2 + zt + z^2 = 0.$$

The quadratic formula gives

$$t = z\frac{-1 \pm 3i}{2},$$

where $i = \sqrt{-1}$ is the imaginary unit; thus, the numbers x, y, and z satisfying the assumptions of the problem cannot all be real. The original formulation of the problem gives no hint that these numbers might have to be complex.

The numbers

$$\zeta_1 = \frac{-1+3i}{2} \quad \text{and} \quad \zeta_2 = \frac{-1-3i}{2}$$

are the primitive cube roots of unity. We have $\zeta_1^3 = \zeta_2^3 = 1$, $\zeta_1^2 = \zeta_2$. and $\zeta_2^2 = \zeta_1$; the last one of these equations follows from the previous ones, since

$$\zeta_2^2 = (\zeta_1^2)^2 = \zeta_1^4 = \zeta_1^3 \cdot \zeta_1 = 1 \cdot \zeta_1 = \zeta_1.$$

5) (JUNIOR 5) Given the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, how many ways can you select seven distinct elements whose sum is divisible by 3? (In other words, how many seven-element subsets does this set have such that the sum of its elements is divisible by 3?)

Source: Dániel Arany Mathematics Competition for Students, 2017/2018, First Round, Beginners Category I-II, Problem 2. See

`https://www.bolyai.hu/files/AD_2017-2018-feladatok_megoldasok.pdf`

Solution: Since the sum $1+2+3+4+5+6+7+8+9$ is divisible by 3, each 7-element subset can be replaced by its complement, and we can ask the simpler question, how many 2-element subsets does the above set have the sum of whose elements is divisible by 3. The remainders of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ when divided by 3 are $1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0$. The question can then be modified to ask how may ways can one pick two of these remainders (considered as distinct items, since they correspond to different numbers) in such a way that their sum is divisible by 3. The only way to do this is to pick two 0s or to pick one 1 and one 2.

The two 0s out of the three 0s can be picked $\binom{3}{2} = 3$ ways. Picking one 1 out of the three 1s and then picking one 2 out of the three 2s can be done in $3 \cdot 3 = 9$ ways. Altogether, this gives $3 + 9 = 12$ ways. That is, one can pick 12 seven element subsets of the above set such that the sum of its elements is divisible by 3.

6) (JUNIOR 6) A graph is a set of objects (called points or vertices) and a set of (unordered) pairs of these vertices; these pairs are called edges. A graph is called *finite* if the set of its vertices is finite. A graph is called *connected* if from any vertex one can get to any other vertex by "traversing" edges. A *neighbor* of a vertex is any vertex connected to it by an edge.

Given a finite connected graph, assign a real number to each of its vertices in such a way that the assigned number is the arithmetic mean of the numbers assigned to its neighbors. Show that the only way this is possible if we assign the same number to all the vertices.

Source: Problem 467, proposed by Dénes König; p. 218, Középiskolai Matematikai és Fizikai Lapok, No. 7, Vol. V, March 1929. See

`http://db.komal.hu/scan/1929/03/92903218.g4.png`

Solution: Let α be the largest number assigned, and let S be the set of vertices to which α is assigned. If $x \in S$, then any neighbor of S must also belong to S, since a number $< \alpha$ cannot be assigned to any neighbor of x. Since the graph is connected, S must contain all its vertices, showing that all the vertices must be assigned the same number α.

7) (JUNIOR 7) Solve the equation

$$\frac{1+x^4}{(1+x)^4} = \frac{1}{2}.$$

Source: Problem 469, p. 218, Középiskolai Matematikai és Fizikai Lapok, No. 7, Vol. V, March 1929. See

`http://db.komal.hu/scan/1929/03/92903218.g4.png`

Solution: The equation can be written as

$$2(1+x^4) = (1+x)^4.$$

Since $x = 0$ is not a solution of this equation, we are free to divide by x. Dividing both sides by x^2, we obtain

$$2\left(\frac{1}{x^2} + x^2\right) = \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)^4;$$

if $x < 0$, then $\sqrt{x}$ is imaginary; however, the calculation is still valid.[1] Noting that

$$\left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)^2 = \frac{1}{x} + x + 2,$$

the equation becomes[2]

$$2\left(\frac{1}{x^2} + x^2\right) = \left(\frac{1}{x} + x + 2\right)^2;$$

Using the identity

$$\left(\frac{1}{x} + x\right)^2 = \frac{1}{x^2} + x^2 + 2,$$

on the left-hand side, and expanding the square on the right-hand side, the last equation becomes

$$2\left(\left(\frac{1}{x} + x\right)^2 - 2\right) = \left(\frac{1}{x} + x\right)^2 + 2 \cdot 2\left(\frac{1}{x} + x\right) + 4,$$

This can be simplified to

$$\left(\frac{1}{x} + x\right)^2 - 4\left(\frac{1}{x} + x\right) - 8 = 0.$$

This is a quadratic equation for $1/x + x$ that is easily solved:

$$\frac{1}{x} + x = \frac{4 \pm \sqrt{4^2 + 4 \cdot 8}}{2} = 2 \pm 2\sqrt{3}.$$

[1] In fact, it will turn out that some solutions of the above equation are complex.

[2] While $\sqrt{x}$ has two values, this identity relies on the requirement that the same value should be assigned to both occurrences of $\sqrt{x}$.

This gives a quadratic equation for x:

$$x^2 - (2 \pm 2\sqrt{3})x + 1 = 0.$$

Solving this for x, we obtain

$$\begin{aligned} x &= \frac{2 + 2\sqrt{3} \pm \sqrt{16 + 8\sqrt{3} - 4}}{2} = \frac{2 + 2\sqrt{3} \pm \sqrt{12 + 8\sqrt{3}}}{2} \\ &= 1 + \sqrt{3} \pm \sqrt{3 + 2\sqrt{3}}, \end{aligned}$$

or

$$\begin{aligned} x &= \frac{2 - 2\sqrt{3} \pm \sqrt{16 - 8\sqrt{3} - 4}}{2} = \frac{2 - 2\sqrt{3} \pm \sqrt{12 - 8\sqrt{3}}}{2} \\ &= 1 - \sqrt{3} \pm \sqrt{3 - 2\sqrt{3}} = 1 - \sqrt{3} \pm i\sqrt{-3 + 2\sqrt{3}}, \end{aligned}$$

where $i = \sqrt{-1}$ is the imaginary unit. The first two solutions are real, the third and fourth are complex.