Junior Prize Exam
Spring 2023

1) Show that if the roots of the equation

$$x^2 + px + q = 0$$

with real coefficients are real, then the roots of the equation

$$x^2 + px + q + (x + a)(2x + p) = 0$$

are also real for all real values of a.

2) Find all prime numbers p such that $p^2 + 2$ is also a prime.

3) Let $n \ge 1$ be an odd integer, and let a_1, a_2, ..., a_n be an arbitrary rearrangement of the numbers 1, 2, ..., n. Show that the product

$$(a_1 - 1)(a_2 - 2) \dots (a_n - n)$$

is even.

4) Let x, y, and z be complex numbers such that

$$x + y + z = 0$$

and

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

Show that then

$$x^3 = y^3 = z^3.$$

5) Given the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, how many ways can you select seven distinct elements whose sum is divisible by 3? (In other words, how many seven-element subsets does this set have such that the sum of its elements is divisible by 3?)

6) A graph is a set of objects (called points or vertices) and a set of (unordered) pairs of these vertices; these pairs are called edges. A graph is called *finite* if the set of its vertices is finite. A graph is called *connected* if from any vertex one can get to any other vertex by "traversing" edges. A *neighbor* of a vertex is any vertex connected to it by an edge.

Given a finite connected graph, assign a real number to each of its vertices in such a way that the assigned number is the arithmetic mean of the numbers assigned to its neighbors. Show that the only way this is possible if we assign the same number to all the vertices.

7) Solve the equation

$$\frac{1 + x^4}{(1 + x)^4} = \frac{1}{2}.$$

Soon after the exam, solutions will appear on the Web Site
http://www.sci.brooklyn.cuny.edu/~mate/prize/2023/

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.