All Problems on the Prize Exams
Spring 2024
Version Date: Sat Mar 9 12:29:25 EST 2024

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Find all pairs of positive integers (x, y) such that $x^2 + y^2 = 17^2$.

Solution: Of course it is possible to test all pairs of integers (x, y) with $0 < x, y < 17$ to see which ones work; however, it is possible to cut down on the number of pairs that need to be tested.

First, note that one of x and y must be even, the other one must be odd. Without loss of generality, we may assume that x is even. We have

$$y^2 = 17^2 - x^2 = (17 - x)(17 + x).$$

Assume p is a common prime factor of $17 - x$ and $17 + x$; then p must have a divisor of $(17 - x) + (17 + x) = 2 \cdot 17$; since x is even, we cannot have $p = 2$; we cannot have $p = 17$, either, since 17 cannot be a divisor of y^2, as $0 < y < 17$. That is, $17 - x$ and $17 - x$ must be relatively prime. Since their product is a square, both of them must be squares. This is because in a square, all prime factors must occur with an even exponent; since $17 - x$ and $17 + x$ have no common prime factors, each separately must be a product of even powers of primes. That is, there are positive odd integers a and b such that $a^2 = 17 - x$ and $b^2 = 17 + x$; that is,

$$a^2 + b^2 = 34;$$

note that we must also have $a < b$ since $x > 0$. Here it is quite easy to test all possibilities for a and b, and we obtain the solutions $a = 3$ and $b = 5$

This leads to the solution $x = 17 - 3^2 = 8$ and $y^2 = 17^2 - 8^2 = 15^2$, that is $x = 8$ and $y = 15$; this is the only solution for x, y with x even. The other solution is $x = 15$ and $y = 8$.

2) (JUNIOR 2 and SENIOR 2) In a manuscript of 60 sheets, the pages are numbered by the numbers $1, 2, 3, \ldots, 120$ in the usual way. Unfortunately, some sheets were lost. The sum of the page numbers on the remaining pages is 7159. How many sheets were lost?

Source: Problem 1, Dániel Arany Mathematics Competition, academic year 2012/2013, Beginners Category I, round 3 (final). See p. 9,

`https://matek.fazekas.hu/index.php?option=com_content&view= article&id=57:arany -daniel-matematikaverseny&catid=26&Itemid=185`

Solution: The sum of the page numbers on all of the pages would be

$$\sum_{i=1}^{120} i = \frac{120 \cdot 121}{2} = 7260.$$

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

Thus, the page numbers on the missing pages add up to $7260 - 7159 = 101 = 4 \cdot 26 - 3$.

The page numbers on the kth page are $2k - 1$ and $2k$; these add up to $4k - 1$. For these to add up to $4 \cdot 26 - 3$ one needs $4l + 3$ sheets for $l = 0, 1, 2, 3 \ldots$, that is 3, 7, 11, ... distinct sheets. However, even on the first 7 sheets the page numbers add up to

$$4 \cdot (1 + 2 + 3 + 4 + 5 + 6 + 7) - 7 = 4 \cdot \frac{7 \cdot 8}{2} - 7 = 105,$$

so 7 or more sheets could not have been lost. So exactly 3 sheets were lost.

3) (JUNIOR 3 and SENIOR 3) Given a 5×5 matrix (a list of numbers, called entries, arranged in a rectangle with 5 rows and 5 columns), with each of the entries being 1 or -1. Form the products of the entries on each of the rows and each of the columns, obtaining 10 products altogether. Show that the sum of all these products cannot be 0.

Source: 41th László Kalmár Mathematics competition for 8th grade students, county round, 2012, Problem 5. See

`https://www.kalmarverseny.hu/2012/07/30/kalmar-laszlo-matematikaverseny-feladat sorok-es-megoldasaik-2012/`

Solution: Each of the row products and each of the column products is ± 1. Let a be the number of row products that are -1, and let b be the number of column products that are -1. The product of the row products and the product of the column products is equal to the product of all entries of the matrix. Hence $(-1)^a = (-1)^b$; that is, a and b have the same parity - thus $a + b$ is even. The sum of all row products plus the sum of all column products is

$$(-a) + (5 - a) + (-b) + (5 - a) = 10 - 2(a + b).$$

As $a + b$ is even, the right-hand side is not divisible by 4; hence it cannot be 0, which is what we wanted to prove.

4) (JUNIOR 4) Given a triangle such that all its altitudes are at least 1. Show that its area is at least $1/\sqrt{3}$.

Source: 47th László Kalmár Mathematics competition for 8th grade students, National final, November 6, 2020, Problem 4. See

`https://www.kalmarverseny.hu/2020/07/30/kalmar-laszlo-matematikaverseny-feladat sorok-es-megoldasaik-2020/`

First Solution: Let T be a triangle satisfying the requirements, write t for its area, a, b, c for its sides, and h_a, h_b, h_c for the altitudes corresponding to these sides in turn. We have

$$2t = ah_a = bh_b = ch_c.$$

Let T' be a triangle with sides

$$a' = \frac{a}{2t} = \frac{1}{h_a}, \quad b' = \frac{b}{2t} = \frac{1}{h_b}, \quad \text{and} \quad c' = \frac{c}{2t} = \frac{1}{h_b}.$$

T' is a triangle similar to T, its linear size being $1/(2t)$ times that of T, so its area is $1/(2t)^2$ times that of the area of T; that is, its area is $1/(4t)$. Since the only condition of T is that its altitudes be ≥ 1, the only condition on T' is that its sides be ≤ 1. Under these conditions, we have to find the minimum value of t, that is, the maximum value of the area of T', i.e., of $1/(4t)$.

So, we can reformulate the question as follows: given a triangle with sides $a', b', c' \leq 1$, what is its maximum area. In doing so, without loss of generality we may assume that $a' = 1$. Indeed, if all sides of the triangle are strictly less than 1, then we may increase each side with the same factor so as to make the largest one equal to 1; we may assume that this side is the side a'. Then look for the triangle for which the altitude h'_a erected on the side a' is the largest possible. This clearly happens when we also have $b' = c' = 1$. om which case $h'_a = \sqrt{3}/2$ giving the maximum area of T': $1/(4t) = \sqrt{3}/4$. That is, the minimum value of t is $1/\sqrt{3}$, as we wanted to show.

Second Solution: First note that in the equilateral triangle with sides $2/\sqrt{3}$, all altitudes have length 1, and its area is $1/\sqrt{3}$.

Assume that $\triangle ABC$ is a triangle with all its altitudes ≥ 1 that has the least possible area. Then one of the altitudes must be exactly 1; indeed, if all altitudes were strictly greater than 1, then the triangle could be replaced with a smaller similar triangle in which one altitude is exactly 1.

Without loss of generality, we may assume that the altitude h_c incident to the vertex C length 1, and further, that $\overline{AC} \leq \overline{BC}$. Placing the triangle in the coordinate system, we may assume that A has coordinates $(0, 0)$, B has coordinates $(b, 0)$ for some $b > 0$. Assume that C is above the x-axis and the altitude h_c is 1. Then C must have y-coordinate 1; writing u for the x-coordinate of C, we must have $b \geq 2u$ since $\overline{AC} \leq \overline{BC}$. Finally, so as to have $h_a \geq 1$ for the altitude incident to A, the line CB must not intersect the circle of radius 1 centered at $A(0, 0)$ (so we must have $u > 0$). For the triangle having the least possible area, $\overline{AB}$ must be as small as possible, since the area of the triangle is $\overline{AB} \cdot h_c/2 = \overline{AB}/2$. If $\overline{BC}$ is not tangent to this circle (in the first quadrant), we may move this line to the left parallel to its original position until it becomes tangent to the circle, since thereby the side $\overline{AB}$ would decrease. After this move, we will have $h_a = 1$. Since such a move would decrease u and b, the x-coordinates of C and B, respectively, by the same amount, if we started out with $b > 2u$, this inequality will also be true for the new values of b and u (note that we must have $u > 0$, as we remarked above, since BC does not enter the inside of the circle in question, and the y-coordinate of C is 1, the radius of the circle).

After this point, we can decrease $\overline{AB}$ by moving B to the left and C to the right while keeping $\overline{BC}$ tangent to the circle; This will increase u and decrease b. However, we must stop when we have $b = 2u$. This is important, since if we moved past this point and we would end up with $b < 2u$, and then we would have $\overline{AC} > \overline{BC}$. Then, since we still have $h_a = 1$ and we have $\overline{AC} \cdot h_b = \overline{BC} \cdot h_a$, since both sides are equal to twice the area of $\triangle ABC$, we would then have

$$h_b = \frac{\overline{BC}}{\overline{AC}} h_a < 1,$$

contrary to the requirement that all three altitudes must be at least 1. The optimal case gives $\overline{AC} = \overline{BC}$, when $h_b = h_a = h_c = 1$, in which case $\triangle ABC$ is equilateral, and $\overline{AB} = \overline{BC} = \overline{AC} = 2/\sqrt{3}$, and the area of $\triangle ABC$ is $1/\sqrt{3}$.

5) (JUNIOR 5) Given 50 positive integers whose sum is at least 100, show that it is possible to choose 3 of them whose sum is at least 6.

Source: Hungarian Mathematics Highschool competitions. Fazekas Gymnasium, 7c, segment III, round 4. It can be located in the file II4.HTM after unzipping the file MATVERS.ZIP at

`http://www.mek.iif.hu/porta/szint/termesz/matemat/matvers/`

In Hungarian.

Solution: Let the numbers be $a_1, a_2, \ldots, a_{50}$, Assume that for any i, j, k with $1 \leq i < j < l \leq 50$

we have $a_i + a_j + a_k \leq 5$. Summing for all such triples we then have

$$\sum_{i,j,k:1\leq i<j<k\leq 50} (a_i + a_j + a_k) \leq 5 \cdot \binom{50}{3}; \tag{1}$$

the binomial coefficient on the right-hand side is the number of triples summed. In this sum, for each l with $1 \leq l \leq 50$, a_l occurs $\binom{49}{2}$ times. That is, the sum onf the left-hand side of (1) equals

$$\binom{49}{2} \sum_{l=1}^{50} a_l \geq 100 \binom{49}{2}. \tag{2}$$

This is, however impossible, since the right-hand side of (2) is smaller than that of (1). Indeed,

$$5 \cdot \binom{50}{3} = 5 \cdot \frac{50 \cdot 49 \cdot 48}{6} = \frac{125}{3} \cdot \frac{49 \cdot 48}{2} = \frac{125}{3} \cdot \binom{49}{2} < 100 \cdot \binom{49}{2}.$$

The proof is complete.

6) (JUNIOR 6) At a dinner party, there are 6 guests, among whom everybody knows at least one other guest (knowing another is symmetric, so if A knows B, then B also knows A). Among the first five guests, each one knows a different number of other guests. How many guests does the sixth guest know?

Source: Based on Problem 3, Dániel Arany Mathematics Competition, academic year 2012/2013, Beginners Category I-II, round I. See p. 2,
`https://matek.fazekas.hu/index.php?option=com_content&view= article&id=57:arany -daniel-matematikaverseny&catid=26&Itemid=185`

Solution: In a more mathematical language, one can talk about a graph of six vertices (points) where the degree (the number of edges incident to a vertex) is different for each of the first five vertices, and each of these degrees is at least 1. This means that the degrees of the first 5 points are $1, 2, 3, 4, 5$, in some order.

Label the vertices as P_1, P_2, P_3, P_4, P_5, and P_6. Write $\deg P$ for the degree of point P. Without loss of generality, we may assume that $\deg P_1 = 5$, $\deg P_2 = 1$, $\deg P_3 = 4$, $\deg P_4 = 2$, $\deg P_5 = 3$. Then P_1 must be connected to each of P_2, P_3, P_4, P_5, P_6. P_2 is already connected to P_1, and it cannot be connected to any other vertex. P_3 must be connected to each of P_4, P_5, P_6. P_4 already connected to P_1 and P_3, and it cannot be connected to any other vertex. Then P_5 must be connected to P_6. This shows that P_6 is connected to P_1, P_3, and P_5. Thus, the degree of P_6 is 3. That is, the sixth guest knows three other guests.

7) (JUNIOR 7) One places dominoes on a 6×6 chess board in such a way that every field is covered and no dominoes overlap. Each domino completely covers two adjacent fields. Show that among the 5 horizontal and 5 vertical lines separating the fields, there is at least one that is not cut by any dominoes.

Source: Problem 3, Országos Középiskolai Tanulmányi Verseny matematikából (National High-school Competition n Mathematics, Hungary) 1962, Category 1, round 2, grades 11-12
`http://versenyvizsga.hu/external/vvszuro/vvszuro.php#2`

Solution: In each column of the chessboard, a vertical domino occupies two fields, so there must also be an even number of fields covered by horizontal dominoes. Therefore each vertical line must be cut by an even number of dominoes. Indeed, in the column to the left of the first vertical line cut by an odd number of dominoes there would be an odd number of fields covered by horizontal dominoes. Thus, to cut all five vertical lines, one needs at least 10 horizontal dominoes.

The same is true also for cutting horizontal lines, so one also needs at least 10 vertical dominoes. However, the total number of dominoes is 18, since there are 36 fields, so there are not enough dominoes to cut all vertical and horizontal lines.

8) (SENIOR 4) In a convex quadrilateral $\square ABCD$ there is an internal P point such that the triangles $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ all have the same area. Show that P is the midpoint of one of the diagonals of the quadrilateral $\square ABCD$.

Source: Hungarian Mathematics Highschool competitions. Fazekas Gymnasium, 7c, segment III, round 7. It can be located in the file II4.HTM after unzipping the file MATVERS.ZIP at

`http://www.mek.iif.hu/porta/szint/termesz/matemat/matvers/`

In Hungarian.

Solution: Write $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ for the vectors $\overrightarrow{PA}$, $\overrightarrow{PB}$, $\overrightarrow{PC}$, and $\overrightarrow{PD}$. The area of the triangle $\triangle ABP$ is half the length of the vector $\mathbf{a}\times\mathbf{b}$; the areas of the other triangles can be written similarly. That is, the equality of the areas of all for triangles is described by the equation

$$\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{d}=\mathbf{d}\times\mathbf{a}\neq 0. \tag{1}$$

As for the last inequality, if it were not true the areas of all the triangles would be 0. Then the rectangle $\square ABCD$ would have area 0, so it would degenerate to a line; so all the points A, B, C, D would be on the same line, since $\square ABCD$ is convex, and P could not be an internal point).

This means that

$$0=\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{b}=(\mathbf{a}+\mathbf{c})\times\mathbf{b}.$$

that is,

$$(\mathbf{a}+\mathbf{c})\times\mathbf{b}=0. \tag{2}$$

Similarly,

$$(\mathbf{a}+\mathbf{c})\times\mathbf{d}=0 \tag{3}$$

(3) implies that $\mathbf{a}+\mathbf{c}\parallel\mathbf{b}$, and (3) implies that $\mathbf{a}+\mathbf{c}\parallel\mathbf{d}$.

There are two possibilities here: 1) $\mathbf{a}+\mathbf{c}=0$, in which case P is the midpoint of the diagonal $\overline{AC}$, and then the assertion of the problem is established; 2) $\mathbf{a}+\mathbf{c}\neq 0$, in which case $\mathbf{b}\parallel\mathbf{d}$. Since $\mathbf{b}\neq 0$ according to the inequality in (1), this means that $\mathbf{d}=\lambda\mathbf{b}$ for some real number λ. Hence, by (1), we have

$$\mathbf{a}\times\mathbf{b}=-\mathbf{a}\times\mathbf{d}=-\mathbf{a}\times(\lambda\mathbf{b})=-\lambda(\mathbf{a}\times\mathbf{b}).$$

Since the vector products here are not zero in view of the inequality in (1), this means that $\lambda=-1$. Hence $\mathbf{d}=-\mathbf{b}$, which means that P is the midpoint of the diagonal $\overline{BD}$, completing the proof.

Remark. If, say, P is the midpoint of the diagonal $\overline{AC}$, then draw two lines b and d parallel to $\overline{AC}$ at equal distance from this line in the two half planes determined by the line. If B then placed on the line b and D is placed on the line d, then the four triangles described in the problem will all have the same area. To satisfy all assumptions of the problem, the points B and D need to be so placed that the quadrilateral $\square ABCD$ be convex.

9) (SENIOR 5) Given a finite commutative group with an odd number of elements, show that the product of all the elements of the group is the identity element.

Source: Problem 3, University of Hyderabad, M.Sc. In Mathematics (Applied Maths) Entrance Exam - Download Previous Years Question Papers, Paper 1, Entrance Examination, 2004, M.Sc. (Mathematics/Applied Mathematics), p. 8.

`http://entrance-exam.net/forum/question-papers/university-hyderabad-m-sc-mathematics-applied-maths-entrance-exam-download-previous-years-question-papers-73542.html`

Solution: Calling the group G, we will show that the only element a of G such that $a^{-1} = a$ is its identity element e. Indeed, let a such an element. Then, denoting by n the number of elements of G and forming the product of all its elements, we have

$$\prod_{g \in G} g = \prod_{g \in G} (ag) = a^n \prod_{g \in G} g = a \prod_{g \in G} g;$$

the first equation holds since when g runs over all elements of G, the products (ag) also run over all elements of G. The last equation holds since n is odd and $a = a^{-1}$, and so $a^2 = e$ (the identity element). Multiplying this equation with the inverse of $\prod_{g \in G} g$ and comparing the sides, it follows that indeed $a = e$, as we claimed.

Now, let $G' \subset G$ be a subset of G such that if $g \in G'$ then $g^{-1} \notin G'$, and let G' be a maximal set with this property. Then

$$G = G' \cup \{g^{-1} : g \in G'\} \cup \{e\}.$$

Hence,

$$\prod_{g \in G} g = e \prod_{g \in G'} (gg^{-1}) = e,$$

as we wanted to show.

10) (SENIOR 6) Let a_n be real numbers for $n \geq 1$. Assume that $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent. Show that then $\sum_{n=1}^{\infty} n^2 a_n$ is divergent.

Source: University of Pennsylvania Mathematics Graduate Preliminary Exam, Fall 2021, Problem 9, p. 19. See

`https://www.math.upenn.edu/graduate/program-description/prelim-exam`

Solution: We will show that under the assumption $\sum_{n=1}^{\infty} n^{\alpha} a_n$ is divergent for any $\alpha > 1$. Indeed, assume, on the contrary, that the latter series is convergent for some $\alpha > 1$. Then $n^{\alpha} a_n \to 0$ as $n \to \infty$; hence, for large enough n, we have $|n^{\alpha} a_n| \leq 1$, and so $|a_n| \leq n^{-\alpha}$. Since

$$\sum_{n=1}^{\infty} n^{-\alpha}$$

is convergent, $\sum_{n=1}^{\infty} |a_n|$ is also convergent, contrary to the assumption, completing the proof.

Note: The original formulation of the problem said the following.

Let $\{a_n\}$ be a sequence of real numbers. For each of the following, give either a proof or a counter example:

(a) If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then $\sum_{n=1}^{\infty} n a_n$ is divergent.

(b) If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then $\sum_{n=1}^{\infty} n^2 a_n$ is divergent.

As we showed above, statement (b) is true. On the other hand, statement (a) is false, as shown by the example[1]

$$a_n = \frac{(-1)^{n+1}}{n \log(n+1)},$$

since

$$\sum_{n=1}^{\infty} \frac{1}{n \log(n+1)}$$

is divergent, as can easily be shown by the integral test, since

$$\int_2^\infty \frac{dx}{x \log x} = \lim_{A \to \infty} \int_2^A \frac{dx}{x \log x} = \lim_{A \to \infty} \int_{\log 2}^{\log A} \frac{dt}{t} = \infty,$$

where the second equation uses the substitution $t = \log x$, when $dt = dx/\log x$. On the other hand,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \log(n+1)}$$

is (conditionally) convergent by the alternating series test.

11) (SENIOR 7) Let f be a differentiable function on the real line. Assume that there is no x such that $f(x) = f'(x) = 0$. Show that the set

$$S = \{x \in [0,1] : f(x) = 0\}.$$

is finite.

Source: Problem 4, Ph.D. Preliminary Examination (Analysis), University of Pittsburgh, August 2008. See
`https://www.mathematics.pitt.edu/graduate/graduate-handbook/sample`
`-preliminary-exams`

Solution: Assume, on the contrary, that the set S is infinite. Then there is an infinite sequence of pairwise distinct elements x_n of S ($1 \le n < \infty$) that is convergent; write $x_0 = \lim_{n\to\infty} x_n$. We may further assume that $x_n \ne x_0$ for any $n \ge 1$ (by removing, if necessary, the single element from this sequence for which $x_n = x_0$). Given that f is differentiable, f is continuous, hence $f(x_0) = 0$; this is because $f(x_n) = 0$, since $x_n \in S$. Now, the derivative of f at x_0 is

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x)}{x - x_0};$$

thus the limit on the right-hand side exists. This limit cannot be anything other than 0, since $x_n \to x_0$, $x_n \ne x_0$ for any $n \ge 1$, and $f(x_n) = 0$. Thus $f(x_0) = f'(x_0) = 0$, contradicting our assumptions.

[1] Here $\log x$ stands for the natural logarithm for x. In mathematics, the notation $\ln x$ is almost never used, and it is common to use $\log x$ to denote the natural logarithm of x. Other sciences often use $\ln x$ to denote the natural logarithm of x.