

SENIOR PRIZE EXAM  
SPRING 2024

- 1) Find all pairs of positive integers  $(x, y)$  such that  $x^2 + y^2 = 17^2$ .
- 2) In a manuscript of 60 sheets, the pages are numbered by the numbers  $1, 2, 3, \dots, 120$  in the usual way. Unfortunately, some sheets were lost. The sum of the page numbers on the remaining pages is 7159. How many sheets were lost?
- 3) Given a  $5 \times 5$  matrix (a list of numbers, called entries, arranged in a rectangle with 5 rows and 5 columns), with each of the entries being 1 or  $-1$ . Form the products of the entries on each of the rows and each of the columns, obtaining 10 products altogether. Show that the sum of all these products cannot be 0.
- 4) In a convex quadrilateral  $\square ABCD$  there is an internal  $P$  point such that the triangles  $\triangle ABP$ ,  $\triangle BCP$ ,  $\triangle CDP$ , and  $\triangle DAP$  all have the same area. Show that  $P$  is the midpoint of one of the diagonals of the quadrilateral  $\square ABCD$ .
- 5) Given a finite commutative group with an odd number of elements, show that the product of all the elements of the group is the identity element.
- 6) Let  $a_n$  be real numbers for  $n \geq 1$ . Assume that  $\sum_{n=1}^{\infty} a_n$  is convergent but not absolutely convergent. Show that then  $\sum_{n=1}^{\infty} n^2 a_n$  is divergent.
- 7) Let  $f$  be a differentiable function on the real line. Assume that there is no  $x$  such that  $f(x) = f'(x) = 0$ . Show that the set

$$S = \{x \in [0, 1] : f(x) = 0\}.$$

is finite.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE  
<http://www.sci.brooklyn.cuny.edu/~mate/prize/2024/>