All Problems on the Prize Exams
Spring 2025
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Let α, β, and γ be such that $\alpha+\beta+\gamma=\pi$. Prove that

$$\frac{\sin\alpha+\sin\beta-\sin\gamma}{\sin\alpha+\sin\beta+\sin\gamma}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}.$$

Source: Problem 880, p. 110, Középiskolai Matematikai Lapok, No. 4, Vol. V, December 1897.
`http://db.komal.hu/scan/1897/12/89712067.g4.png`

Solution: We have

$$\sin\gamma=\sin(\pi-\alpha-\beta)=\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

Hence,

$$\begin{aligned}\frac{\sin\alpha+\sin\beta-\sin\gamma}{\sin\alpha+\sin\beta+\sin\gamma}&=\frac{\sin\alpha(1-\cos\beta)+\sin\beta(1-\cos\alpha)}{\sin\alpha(1+\cos\beta)+\sin\beta(1+\cos\alpha)}\\&=\frac{\frac{1-\cos\beta}{\sin\beta}+\frac{1-\cos\alpha}{\sin\alpha}}{\frac{1+\cos\beta}{\sin\beta}+\frac{1+\cos\alpha}{\sin\alpha}};\end{aligned}$$

the second equation was obtained by dividing the numerator and the denominator by $\sin\alpha\sin\beta$. Using the tangent half-angle formulas (see Note below)

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta},$$

the right-hand side above becomes

$$\frac{\tan(\beta/2)+\tan(\alpha/2)}{\frac{1}{\tan(\beta/2)}+\frac{1}{\tan(\alpha/2)}}=\frac{\big(\tan(\beta/2)+\tan(\alpha/2)\big)\tan(\alpha/2)\tan(\beta/2)}{\tan(\alpha/2)+\tan(\beta/2)}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2};$$

the first equation follows by multiplying the numerator and the denominator by $\tan(\alpha/2)\tan(\beta/2)$. This completes the proof of the equation in the problem.

Note. *The tangent half-angle formulas.*

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

Writing $\eta = \theta/2$, we have

$$\begin{aligned}\tan\frac{\theta}{2} &= \tan\eta = \frac{\sin\eta}{\cos\eta} = \frac{2\sin\eta\cos\eta}{2\cos^2\eta} = \frac{\sin 2\eta}{1+\cos 2\eta} = \frac{\sin\theta}{1+\cos\theta} \\ &= \frac{(1-\cos\theta)\sin\theta}{(1-\cos\theta)(1+\cos\theta)} = \frac{(1-\cos\theta)\sin\theta}{(1-\cos^2\theta)} = \frac{(1-\cos\theta)\sin\theta}{\sin^2\theta} \\ &= \frac{1-\cos\theta}{\sin\theta};\end{aligned}$$

here the fourth equation uses the double-angle formulas for sine and cosine. This establishes the tangent half-angle formulas above.

2) (JUNIOR 2 and SENIOR 2) What is the remainder if 18^{82} is divided by 11?

Solution: According to Fermat's theorem, if p is a prime and $a \nmid p$ for an integer a, then $a^{p-1} \equiv 1 \mod p$. Hence, $18^{10} \equiv 1 \mod 11$. So, $(18^{10})^8 = 18^{80} \equiv 1 \mod 11$. Hence,

$$18^{81} = 18^{80} \cdot 18 \equiv 1 \cdot 7 \equiv 7 \mod 11,$$

and

$$18^{82} = 18^{81} \cdot 18 \equiv 7 \cdot 7 \equiv 49 \equiv 5 \mod 11.$$

3) (JUNIOR 3 and SENIOR 3) Determine a and b in the equation

$$x^3 + ax^2 + bx + 6 = 0$$

such that one of its roots be 2, and another be 3.

Source: Problem 1418, p. 3, Középiskolai Matematikai Lapok, No. 2, Vol. XIII, October 1905.
`http://db.komal.hu/scan/1905/10/90510031.g4.png`

Solution: If the third root is x_3, then the left-hand side of the equation equals

$$(x-2)(x-3)(x-x_3),$$

that is, the constant term of the equation is $(-2)(-3)x_3 = 6x_3$. Since the constant term is given as 6. we have $x_3 = 1$; so the left-hand side of the equation is

$$(x-2)(x-3)(x-x_3) = x^3 - 4x^2 + x + 6,$$

That is, $a = -4$ and $b = 1$.

4) (JUNIOR 4) Determine the value of p in the equation $x^2 - px + 36 = 0$ such that we have

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{5}{12}$$

holds for the roots x_1 and x_2 of the equation.

Source: Problem 1355, Középiskolai Matematikai Lapok, No. 6, Vol. XXI, January 1914.
`http://db.komal.hu/scan/1914/01/91401120.g4.png`

Solution: The roots of the equation

$$x^{-2} - px^{-1} + 36 = 0$$

are $1/x_1$ and $1/x_2$. Multiplying this equation by $x^2/36$, we obtain the equivalent equation

$$x^2 - \frac{p}{36}x + \frac{1}{36} = 0.$$

The sum of the roots of this equation is p; that is

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{p}{36}.$$

Hence, we need to choose $p = 36 \cdot 5/12 = 15$.

5) (JUNIOR 5) Let a, b. amd c be real numbers such that $a^2 + b^2 + c^2 = 1$. Show that

$$-\frac{1}{2} \le ab + bc + ca \le 1.$$

Source: problem 1 17th Eötvos Competition, Hungary, 1910. See
https://imomath.com/index.cgi?page=collectionHun

Solution: We have

$$0 \le (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 1 + 2ab + 2bc + 2ca.$$

Rearranging this, we obtain the first inequality in the question.

Further, we have

$$0 \le (a - b)^2 + (b - c)^2 + (c - a)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ca) = 2(1 - ab - bc - ca).$$

Rearranging this, we obtain the second inequality in the question.

6) (JUNIOR 6) Show that every positive integer that is not divisible by 10 has a multiple that begins and ends with the same digit when written in the decimal system (without starting zeros).

Source: 53rd László Kalmár Mathematics competition for 8th grade students, second day, national final, 2023-2024, Problem 2. See
https://www.kalmarverseny.hu/2024/07/13/kalmar-laszlo-matematikaverseny-feladatgyujtemeny-2013-2024/

Solution: Let n be the number in question, and let k be a positive integer that $n < 10^k$. Let the last digit of n be l. Let m be such that

$$10^{k+2}l \le mn < (10^{k+2} - 10^k.$$

Then, it is easy to see that the first and last digits of $(10m + 1)m$ are both l.

7) (JUNIOR 7) How many ways can you arrange the numbers 0, 1, 2, 3, 4, 5, and 6, in a sequence such that the sum of any four consecutive ones among them is divisible by 3.

Based on Problem 1, Dániel Arany Mathematics Competition, academic year 2012/2013, Beginners Category II, round 3 (final). See p. 12,
https://matek.fazekas.hu/index.php?option=com_content&view= article&id=57:arany -daniel-matematikaverseny&catid=26&Itemid=185

Solution: Let $a_1a_2a_3a_4a_5a_6a_7$ be an arrangement of the above numbers in a sequence satisfying the conditions. Since

$$a_1 + a_2 + a_3 + a_4 \quad \text{and} \quad a_2 + a_3 + a_4 + a_5$$

are both divisible by 3, we must have

$$a_1 \equiv a_5 \mod 3.$$

Similarly, since

$$a_2 + a_3 + a_4 + a_5 \quad \text{and} \quad a_3 + a_4 + a_5 + a_6$$

are both divisible by 3, we must have

$$a_2 \equiv a_6 \mod 3.$$

A similar argument also gives that

$$a_3 \equiv a_7 \mod 3.$$

Hence, we cannot have

$$a_4 \equiv 1 \mod 3$$

or

$$a_4 \equiv 2 \mod 3,$$

since there are only two among that the given numbers that are congruent to $1 \mod 3$, and, similarly, there are only two among that the given numbers that are congruent to $2 \mod 3$. Hence, we must have

$$a_4 \equiv 0 \mod 3.$$

Thus, considering only how many ways the residue classes can be arranged, the first three may be 0, 1, 2 in any order, the last three must follow the same arrangement, and the fourth can only be 0. This gives $3! = 6$ arrangements. The three occurrences of the residue class 0 must be replaced by one of each of the numbers 0, 3, and 6; this can be done in $3! = 6$ ways. The two occurrences of the residue class 1 must be replaced by one of each of the numbers 1 and 4. This can be done in two ways. Similarly, the two occurrences of the residue class 2 must be replaced by one of each of the numbers 2 and 5. This can be done in two ways. That is, altogether, there are $6 \cdot 6 \cdot 2 \cdot 2 = 144$ arrangements.

8) (SENIOR 4) Assume all but one edges of a tetrahedron have length ≤ 1 (no restriction is placed on the length of the remaining one edge). Prove that the volume of the tetrahedron is $\leq 1/8$.

Source: Ninth International Olympiad, 1967, Problem 1967/3. See

`http://www.imo-official.org/year_info.aspx?year=1967`

Solution: Call the vertices of the tetrahedron A, B, C, and D, and assume all edges except BC have length ≤ 1, while no restriction is placed on the length of the edge BC. Set up a coordinate system with the origin at D such that A is on the x-axis, B is in the xy-plane, and all coordinates of these points are nonnegative. That is, the coordinates of the points are $A(x, 0, 0)$, $B(b_1, y, 0)$, and $C(c_1, c_2, z)$ for some nonnegative x, y, z, b_1, c_1, and c_2. The volume of the tetrahedron can be expressed with a determinant:

$$V = \frac{1}{6} \begin{vmatrix} x & 0 & 0 \\ b_1 & y & 0 \\ c_1 & c_2 & z \end{vmatrix} = \frac{1}{6} xyz.$$

That is, we may move around the point B by changing only its first coordinate, and the point C by changing only its first or second coordinates, while making sure that the lengths of the edges AB, DB, AC, and DC will not exceed 1; in this way, we can change the first coordinates of B and C to $x/2$. Indeed, when doing so, the new lengths of BA and BD will not exceed the old length of the larger of BA and BD. Similarly, the new lengths of AC and CD will not exceed the old length of the larger of AC and CD.

So, from now on assume that the coordinates of the points are

$$A(x,0,0), B(x/2,y,0), C(x/2,c_2,z), \quad \text{and} \quad D(0,0,0).$$

The coordinates x, y, z, and c_2 need to satisfy the following inequalities:

$$\begin{aligned} x &\le 1, \\ \left(\frac{x}{2}\right)^2 + y^2 &\le 1, \\ \left(\frac{x}{2}\right)^2 + c_2^2 + z^2 &\le 1. \end{aligned}$$

The first of these inequalities says that $DA \le 1$, the second one, that $DB = BA \le 1$, and the third one, that $DC = CA \le 1$. We weaken the third inequality to

$$\left(\frac{x}{2}\right)^2 + z^2 \le 1$$

Thus

$$y, z \le \sqrt{1 - \frac{x^2}{4}}.$$

Hence, for the volume V of the tetrahedron, we have

$$V = \frac{1}{6}xyz \le \frac{1}{6}x\left(1 - \frac{x^2}{4}\right) = \frac{1}{6}\left(x - \frac{x^3}{4}\right).$$

Write $f(x)$ for the right-hand side. We have $f'(x) = (1/6)(1 - 3x^2/4)$. Therefore, $f'(x) > 0$ when $0 \le x \le 1$, showing that f is increasing on $[0,1]$. So,

$$V \le f(1) = \frac{1}{6} \cdot \frac{3}{4} = \frac{1}{8},$$

which is what we wanted to prove.

9) (SENIOR 5) Let ABC be a triangle such that the coordinates of its vertices are integers, and its area is a positive integer (taking the area of the unit square of the coordinate system to be 1), Show that the midpoint of one of the sides of the triangle has integer coordinates.

Source: 53rd László Kalmár Mathematics competition for 8th grade students, second day, national final, 2023-2024, Problem 3. See

`https://www.kalmarverseny.hu/2024/07/13/kalmar-laszlo-matematikaverseny-feladatgyujtemeny-2013-2024/`

Solution: Without loss of generality, we may assume that A is the origin of the coordinate system, and the points B and C lie in the right half plane. Writing $B(k,l)$ and $C(m,n)$ for the coordinates of the vertices B and C, the latter assumption means that $k, m \geq 0$. We may also assume that $m \leq k$. Let I_1, I_2, and I_3 denote the area between the sides AB, AC, and CB and the x-axis, respectively, with areas under the x-axis counting as negative (the same way as in integrals). These areas are easy to calculate either by integration or by direct geometric considerations. We have

$$I_1 = \frac{kl}{2}, \qquad I_2 = \frac{mn}{2}, \quad \text{and} \quad I_3 = \frac{(k-m)(l+n)}{2}.$$

For example, to calculate I_3, one can use the formula for the area of the trapezoid, or one can use integration:

$$I_3 = \int_m^k \left(\frac{l-n}{k-n}(x-m) + n \right) dx = \frac{(k-m)(l-n)}{2} + (k-m)n = \frac{(k-m)(l+n)}{2}.$$

The area of the triangle is[1]

$$|I_1 - I_2 - I_3| = \frac{1}{2} \left| kl - mn - (k-m)(l+n) \right|$$

In order for this to be an integer, either all of the numbers kl, mn, and $(k-m)(l+n)$ must be even, or two of them must be odd and one even; in other words, either all three integrals (areas) must be integers. or two of them must be nonintegers, and the third one, an integer.

In order for the coordinates of the midpoint of one of the sides be integers, there must be a pair among k, l, or m, n, or $k-m, l+m$ such that both members of the pair are even. Assume this is not the case, that is, each pair contains at least one odd member. We will show that in this case, the area of the triangle cannot be an integer.

First note, that the three pairs have equal roles in the sense that if the first members of two pairs are odd, then the first member of the third pair is even. If one of them is odd, the other is even, then the third one is odd. If two of them are even, then the third one is also even. Similarly for the second members.

Assume first that all three areas I_1, I_2, and I_3 are integers, In this case each pair contains an even integer; for two pairs, these even integers must be the same member. For example, if k and m are even, then l and n must be odd; but then $k-m$ and $l+m$ are both even, so the midpoint of CB has integer coordinates. A similar argument applies in the other cases.

Now assume that the first two areas are not integers. Then k, l, m, and n are odd; but then again $k-m$ and $l+m$ are both even, so the midpoint of CB has integer coordinates. A similar argument applies in the other cases.

This establishes the assertion that the midpoint of one of the sides has integer coordinates.

10) (SENIOR 6) Show that the integral

$$\int_0^{\sqrt{\pi}} \cos x^2 \, dx$$

is positive.

[1]The expression in the next line can be simplified as $|kn - ml|/2$. This can also be obtained from the determinant form of the area of a triangle. In any case, the form of the area we wrote in the next line is more suitable for the argument that follows.

Source: Based on Problem 2. 2018 Siberian mathematical contest, Second-or-more year students. See

`https://drive.google.com/file/d/1IRjKq06xLcjrk2s2_uPhvBqbPrdAZoGc/view`

Solution: We will use the substitution $t = x^2$, $dt = 2x\,dx$, i.e., $dx = dt/(2\sqrt{t})$. This substitution introduces a singularity at $x = 0$, so we are technically dealing with improper integrals; for this reason, we introduce limit at 0. Using $t \searrow 0$ to indicate that t tends to zero from the right (the downward arrow suggests that t needs to decrease to reach 0), we have

$$\int_0^{\sqrt{\pi}} \cos x^2\,dx = \lim_{\epsilon\searrow 0}\int_\epsilon^{\sqrt{\pi}} \cos x^2\,dx = \lim_{\epsilon\searrow 0}\int_{\epsilon^2}^{\pi} \frac{\cos t}{2\sqrt{t}}\,dt = \int_0^{\pi} \frac{\cos t}{2\sqrt{t}}\,dt$$
$$= \int_0^{\pi/2} \frac{\cos t}{2\sqrt{t}}\,dt + \int_{\pi/2}^{\pi} \frac{\cos t}{2\sqrt{t}}\,dt > \int_0^{\pi/2} \frac{\cos t}{2\sqrt{t}}\,dt + \int_{\pi/2}^{\pi} \frac{\cos t}{2\sqrt{\pi/2}}\,dt;$$

the inequality holds since the numerator in the second integral after the fourth equation is negative, and we replaced the denominator with its smallest value in the interval of integration. In the second integral on the right-hand side, we make the substitution $u = \pi - t$, $du = -dt$. So, the right-hand side equals

$$\int_0^{\pi/2} \frac{\cos t}{2\sqrt{t}}\,dt - \int_{\pi/2}^{0} \frac{\cos(\pi - u)}{2\sqrt{\pi/2}}\,du = \int_0^{\pi/2} \frac{\cos t}{2\sqrt{t}}\,dt - \int_0^{\pi/2} \frac{\cos u}{2\sqrt{\pi/2}}\,du$$
$$= \int_0^{\pi/2} \cos t\left(\frac{1}{2\sqrt{t}} - \frac{1}{2\sqrt{\pi/2}}\right)dt > 0;$$

in the first equation, we interchanged the limits of the second integral and used the equality $\cos t = -\cos(\pi - t)$. The inequality on the right holds because the integrand is positive inside the interval of integration. This proves the assertion that the integral in question is indeed positive.

11) (SENIOR 7) Let f be a real-valued function of two real variables such that all second order partial derivatives of f exist and are continuous everywhere. Assume that $f(x, y) = 0$ whenever $xy = 0$. Show that there is a constant $C > 0$ such that

$$|f(x, y)| \le C|x||y| \quad \text{whenever} \quad |x|, |y| \le 1.$$

Source: Stanford University Ph.D. Qualifying Exam in Real Analysis (Mathematics) Problem 1, part I, Fall 2024. See

`https://mathematics.stanford.edu/academics/graduate-students/phd-program/phd-qualifying-exams`

Solution: Let C be a bound for the absolute values of the first and second partial derivatives of f for x and y with $|x|, |y| < 1$. Note that we have $f_x(x, 0) = 0$ for every x for the partial derivative f_x, since $f(x, 0) = 0$ for all x according to the assumptions. Using the Mean-Value Theorem of Differentiation, for x and y with $|x|, |y| < 1$, there is an η between 0 and y such that we have

$$|f_x(x, y)| = |f_x(x, y) - f_x(x, 0)| = |y f_{xy}(x, \eta) \le |y|C.$$

Hence, using the Mean-Value Theorem of Differentiation again, there is a ξ between 0 and x such that

$$|f(x, y)| = |f(x, y) - f(x, 0)| = |x f_x(\xi, y) \le |x||y|C,$$

where the inequality holds according to the previous displayed inequality with $f_x(\xi, y)$ replacing $f_x(x, y)$. This completes the proof.