Section 1.3 - Introduction to Sets

Ari Mermelstein

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1 Set

What is a set? A set is a collection of items. Can be a set of students, numbers, desks, etc. We will (mostly, for now), deal with sets of numbers.

2 Types of Sets

A set can have a finite number of elements, or an infinite number of elements. For example, the set $\{1, 2, 3, 4\}$ is a finite set. The set $\{1, 2, 3 \cdots\}$ is an infinite set.

3 Important Sets in Mathematics

Some sets come up in mathematics and computer science so often, that be gave them special names. Examples:

- $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ =The set of natural numbers.
  $52 \in \mathbb{N}$ but $-3 \notin \mathbb{N}$.
- $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ =The set of integers.
  $-100 \in \mathbb{Z}$ but $\frac{-3}{4} \notin \mathbb{Z}$.
- $\mathbb{Z}^+ = \mathbb{P} = \{1, 2, 3, \cdots\}$ =The set of positive integers.
  $0 \notin \mathbb{Z}^+$.
- $\mathbb{Z}^- = \{-1, -2, -3, \cdots\}$ =The set of negative numbers
- $\mathbb{Q} = \{0, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{11111}{12345}\}$ = The set of rational numbers.
- $\mathbb{R}$ =The set of real numbers.
  $\pi, e, \sqrt{2} \in \mathbb{R}$, but $\sqrt{-1} = i \notin \mathbb{R}$.

To say that a number x is an element of a set S, we write $x \in S$. If a different number, say y is not an element of S, then we write $y \notin S$.

4 A different way of describing sets

We can describe sets using set builder notation. We can write something like:

$$T = \{n \in \mathbb{N} \mid n \leq 6\} = \{0, 1, 2, 3, 4, 5, 6\}.$$

or

$$E = \{z \in \mathbb{Z} \ : \ z = 2k \text{ for some } k \in \mathbb{Z}\} = \{z \in \mathbb{Z} : z \text{ is an even number}\}.$$

or

$$V = \{x \in \mathbb{R} \mid x \geq 1\} = [1, \infty).$$

or

$$\mathbb{Q} = \{q \mid x = \frac{a}{b} \text{for some } a, b \in \mathbb{Z}\}.$$

5 Definitions

Definition: A set S is a subset of a set T if every element of S is also an element of T. We write this as $S \subseteq T$. In other words, for every $x \in S, x \in T$.

TODO: examples

Definition: Suppose S and T are sets. Then we say that $S = T$ if $S \subseteq T$ and $T \subseteq S$.

Definition: We say that S is a proper subset of T if $S \subseteq T$ and $S \neq T$. This is sometimes written as $S \subset T$. (I hate that notation!)

6 The empty set

Take a look at these:

$\{n \in \mathbb{N} \mid 2 < n < 3\}$ or $\{x \in [0, 1] \mid x = \pi\}$.

What do you notice? They have no elements whatsoever! We denote this as $\emptyset$.

Consider this peculiarity: is $\emptyset \subseteq \{1, 2, 3, 4\}$? Oddly, yes.

In fact, $\emptyset \subseteq S$ for any set S in the whole entire world!

7 Cardinality

Definition: The cardinality of a set S, denoted $|S|$ is the number of elements S contains.

Example: $|\{1, 2, 3, 4\}| = 4$, $|\emptyset| = 0$, $|\mathbb{N}| = \cdots$. (Stay tuned!)

8 Power set

Definition: Let S be a set. Then the power set of S, denoted $\mathcal{P}(S)$ is the set that contains all possible subsets of S. In other words, $\mathcal{P}(S) = \{T \mid T \subseteq S\}$.

Example: Let $S = \{1, 2, 3\}$. Then $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.

Theorem 1 (Size of a power set). *If $|S| = n$, then $|\mathcal{P}(S)| = 2^n$.*

Proof. Stay tuned! □

9 Alphabets and Languages

Definition: A set Σ is called an alphabet if $|\Sigma| > 0$ and whose elements are considered symbols. Each symbol $a \in \Sigma$ is called a letter.

Definition: A word is a finite sequence of letters from Σ. The unique word that has no letters is denoted either λ or ϵ.

Definition: A language is a set of words. The language that represents the set of all possible words under an alphabet Σ is denoted Σ^*.

Definition: The concatenation of 2 words $w, x \in \Sigma^*$, denoted wx is a new word z whose first part is w and whose second part is x. When used in the context of concatenation, an exponent or a star can be used. For example, $(ab)^2 = abab$ and $(ab)^*$ means ab repeated 0 or more times.

Example: Let $\Sigma = \{a, b, c\}$. Give an example of a word. Answers: $a, aa, aaa, ab, ac, ca, \epsilon$,etc.

Question: Give an example of a language. Answers: $\{w \mid w = 0^n 1^n$ for some $n \in \mathbb{N}\}, \emptyset, \{w \mid w = 01^*0\}$, etc.

Example: What is Σ^*? Answer: $\{\epsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, aac, aba, abb, abc,$ etc.$\}$.

Examples:

1. List 5 elements of the following sets:
   (a) $\{n \in \mathbb{N} :$ n is divisible by 5$\}$.
   (b) $\{2n + 1 : n \in \mathbb{P}\}$.
   (c) $\mathcal{P}(\{1, 2, 3, 4, 5\})$.
   (d) $\{2^n : n \in \mathbb{N}\}$.

2. List all elements in the following sets:

   (a) $\{n \in \mathbb{N} : n^2 = 9\}$.
   (b) $\{n \in \mathbb{Z} : n^2 = 9\}$.
   (c) $\{x \in \mathbb{Z} : x^2 = 3\}$.

3. How many elements are in $\{x \in \mathbb{Q} : 0 \leq x \leq 73\}$?