Section 1.4 - Set Operations

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1 Basic Set Operations

1.1 Union

Definition: The union of 2 sets A and B, denoted $A \cup B$, is a new set that contains all elements that are in A or B (or both). i.e. $A \cup B = \{x : x \in A or x \in B\}$.

1.2 Intersection

Definition: The intersection of 2 sets A and B, denoted $A \cap B$, is a new set that contains all elements that are in A and B. i.e. $A \cap B = \{x : x \in A and x \in B\}$.

Definition: A and B are said to be disjoint if $A \cap B = \emptyset$.

1.3 Examples

1. **Example:** Let $A = \{n \in \mathbb{N} : n \leq 11\}, B = \{n : n$n is even and $n \leq 20\}, E = \{n :$ n is even$\}$.

   Find $A \cup B, A \cup E, A \cap B$.

2. **Example:** Let $\Sigma = \{a, b\}, A = \{\lambda, a, aa, aaa\}, B = \{\lambda, b, bb, bbb\}$ and $C = \{w \in \Sigma^* :$ length(w)=2$\}$.

   Find All.

1.4 Relative Complement

Definition: The relative complement of one set A and another set B, denoted $A \backslash B$ is a new set that contains the elements that are in A but not in B. i.e. $A \backslash B = \{x : x \in A$ and $x \notin B\}$. You can think of this operation as "minus."

1.5 Symmetric Difference

Definition: The symmetric difference of 2 sets A and B, denoted $A \oplus B$, is a new set that contains all elements in A or B but not both.

Theorem 1 (alternative definition of symmetric difference). $A \oplus B = (A \cup B) \backslash (A \cap B)$.

1.6 Examples

1. Use 1) from above:

   Find:

   (a) $E \backslash B$
   (b) $A \backslash B$

(c) $A \oplus B$

2. Use 2) from above: Find

   (a) $A \backslash B$

   (b) $A \backslash \Sigma$

2 Venn Diagrams

TODO: Draw some on the board

2.1 Absolute Complement

Usually, when we talk about sets, we consider (often implicitly) a universal set, U. This may be $\mathbb{R}$ or $\mathbb{N}$ or a set of vectors, etc.

Based on this understanding, we can define the (absolute) complement of a set.

Definition: The absolute complement of a set A, denoted as A^c or $\overline{A}$ is the set of things under consideration that are not in A. i.e. $\overline{A} = \{x : x \in U \text{ and } x \notin A\}$.

Theorem 2 (Alternative Definition of Relative Complement)**.** $A \backslash B = A \cap B^c$.

2.2 Examples

1. Let $U = \mathbb{N}$ and look a example 1) Find:

   (a) A^c

   (b) E^c

2. Let $U = \mathbb{R}$. Find $[0, 1]^c$.

3 Some Laws of Sets (We will prove some of these eventually)

- $A \cup B = B \cup A \quad A \cap B = B \cap A$ Commutative
- $A \cup (B \cup C) = (A \cup B) \cup C \quad A \cap (B \cap C) = (A \cap B) \cap C$ Associative
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Distributive law
- $A \cup A = A \cap A = A$ Idempotent (meaning, unchanged laws
- $A \cup \emptyset = A \quad A \cap \emptyset = \emptyset \quad A \cup U = U \quad A \cap U = A$. Identity Laws
- $(A^c)^c = A \quad A \cup A^c = U \quad A \cap A^c = \emptyset$. double complementation
- $(A \cup B)^c = A^c \cap B^c \quad (A \cap B)^c = A^c \cup B^c$. De Morgan's laws

4 Some proofs

Theorem 3 (De Morgan's law)**.** $(A \cap B)^c = A^c \cup B^c$

Proof. Proof $\Rightarrow$: Suppose $x \in (A \cap B)^c$. Then, $x \notin (A \cap B)$. Therefore, $x \notin A$ or $x \notin B$. (Since if x were in the intersection, it would be in both A and B). Consequently, $x \in A^c$ or $x \in B^c$. Hence, $x \in (A^c \cup B^c)$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

Proof $\Leftarrow$: Suppose $x \in (A^c \cup B^c)$. So $x \in A^c$ or $x \in B^c$. So $x \notin A$ or $x \notin B$. Therefore, $x \notin (A \cap B)$. (Because if it were, then it would have to be in A and B). Hence,$x \in (A \cap B)^c$. This shows that $A^c \cup B^c \subseteq (A \cap B)^c$.

$\square$

Theorem 4 (distributive law)**.** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. Proof $\Rightarrow$: Suppose $x \in A \cup (B \cap C)$. So then $x \in A$ or $x \in (B \cap C)$. So $x \in A$ or $x \in B$ and $x \in C$.

Suppose $x \in A$. Then $x \in (A \cup B)$ and $x \in (A \cup C)$. Then $x \in (A \cup B) \cap (A \cup C)$. If $x \notin A$. Then $x \in B$ and $x \in C$. Then $x \in (A \cup B)$ and $x \in (A \cup C)$. Then $x \in (A \cup B) \cap (A \cup C)$.

Proof $\Leftarrow$: Suppose $x \in ((A \cup B) \cap (A \cup C))$. So $x \in (A \cup B)$ and $x \in (A \cup C)$. So So ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$). Suppose $x \in A$. Then $x \in (A \cup (B \cap C))$. If $x \notin A$, then $x \in B$ and $x \in C$, so $x \in (B \cap C)$, and $x \in (A \cup (B \cap C))$.

□

5 Cartesian Product

Definition: Consider two sets S and T. The Cartesian product of S and T, denoted $S \times T$ is a set of ordered pairs. In each ordered pair, the first component is from S, and the second component is from T. i.e. $S \times T = \{(s,t) : s \in S \text{ and } t \in T\}$.

Example: $A = \{1,2,3\}, B = \{4,5,6\}$. Then $A \times B = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$.

Definition: Sometimes $S \times S$ is denoted S^2.

Definition: Suppose we have a collection of sets $S_1, S_2, S_3, \cdots S_n$. Then $S_1 \times S_2 \times S_3 \cdots \times S_n = \{(s_1, s_2, s_3, \cdots s_n) : s_i \in S_i \text{ for } i \in \{1,2,3,\cdots,n\}\}$.