Section 1.7 - Properties of Functions

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1 Informal Definitions of Terms

Suppose we had 2 sets $S = \{$s : s is a student in this class$\} and$D$= \{d : d$ is a desk in this room$\}$. We define a function $f : S \to D$ such that $f(s) = d$ if student s is assigned to sit in chair d. If no 2 students are assigned the same chair, we say that f is one-to-one. If every chair has at least one student assigned to it, then we say that f is onto (or more verbosely, f maps S onto T). If F is both one-to-one and onto, we say that f is a one-to-one correspondence.

2 Formal definitions

2.1 One-to-One

Definition: Suppose $f : S \to T$. We say that f is one-to-one if each element of S has a unique image in T. Equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2 \ \forall x_1.x_2 \in S$.

2.2 Onto

Definition: Suppose $f : S \to T$. We say that f is onto if $Im(f) = T$. In other words, for every $t \in T$, there is at least one $s \in S$ such that $f(s) = t$.

2.3 One-to-One Correspondence

Definition: Suppose $f : S \to T$. We say that f is a one-to-one correspondence if f is both one-to-one and onto.

3 Examples

Is the function one-to-one? onto?

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x - 5$.
2. $g : \mathbb{N} \to \mathbb{N}, f(n) = 2n$.
3. $length : \Sigma^* \to \mathbb{N}$.
4. Suppose S is a set and $A \subseteq S$. $\chi_A : S \to \{0, 1\}$.
5. $h : \mathbb{R} \to (0, \infty), f(x) = e^x$.
6. $j : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}, j(x) = \frac{1}{x}$.

4 Inverse Functions

4.1 Definition

The inverse of a function $f : S \to T$ is a function $f^{-1} : T \to S$ such that $f^{-1}(f(x)) = x \ \forall x \in S$ and $f(f^{-1}(y)) = y \ \forall y \in T$.

4.2 Theorem

Theorem 1 (Function Inverse Theorem)**.** *A function f is invertible, (i.e. it has an inverse) if and only if f is a one-to-one correspondence.*

4.3 Examples

For the above functions that are one-to-one correspondences, find the inverse.

5 Image and Pre-image

5.1 Image

Definition: Consider a function $f : S \to T$ and $A \subseteq S$. Then the image of A under f, denoted $f(A) = \{f(x) : x \in A\}$.

5.2 Pre-image

Definition: Consider a function $f : S \to T$ and $B \subseteq T$. Then the pre-image of B under f, denoted $f^{\leftarrow}(B) = \{x \in S : f(x) \in B\}$. We can also talk about the pre-image of a particular element $y \in B$.

Note: the pre-image of a function is equal to the inverse function only if the original function is a one-to-one correspondence. Otherwise, the pre-image is a set rather than one value.

5.3 Examples

Find the pre-image for the following:

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$. Find $f((1, 2)), f^{\leftarrow}(4), f^{\leftarrow}([1, 9])$
2. $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, g(m, n) = m^2 + n^2$. Find $f^{\leftarrow}(0), f^{\leftarrow}(3)$.
3. $length : \Sigma^* \to \mathbb{N}$. Find the pre-image of 2 if $\Sigma = \{a, b\}$.