Section 13.1 - Predicate Calculus

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1 Predicate Calculus

1.1 Definition

Definition: The predicate calculus is a logical system which uses the symbols from propositional logic but augments it using the quantifiers $\exists$ and $\forall$.

Recall: Quantifiers are applied to families of propositions of the form $\{p(x) : x \in U\}$ where U is the universe of discourse.

Recall: The compound proposition $\forall x, p(x)$ is true when $p(x)$ is true for every choice of $x \in U$. Otherwise, $\forall x, p(x)$ is false.

Recall:The compound proposition $\exists x, p(x)$ is true if $p(x)$ is true for at least one $x \in U$. $\exists x, p(x)$ is false if $p(x)$ is false for all $x \in U$.

Recall: We need to specify the universe of discourse, or else statements using quantifiers are meaningless.

2 Multiple Variables

2.1 Examples

We can come up with examples that have more than one variable. Example: Let $BP(x, y)$ means that x is a biological parent of y.

What do the following mean?

1. $\forall x, BP(x, y)$
2. $\forall y, BP(x, y)$
3. $\exists x, BP(x, y)$
4. $\exists y, BP(x, y)$

Note that for some of these, you can't tell what the truth value is until you bind the other variable.

1. $\forall x \forall y BP(x, y)$
2. $\forall x, \exists y BP(x, y)$
3. $\exists x, \forall y BP(x, y)$
4. $\forall y, \exists x BP(x, y)$.
5. $\exists y, \exists x, BP(x, y)$

3 Bound and Free Variables

Definition: Consider the predicate $p(x)$. x is considered a free variable, because as we let x take on different values of U, p(x) will have a different truth value. If we instead consider something of the form $\exists x p(x)$ or $\forall x p(x)$ then x is a bound variable since x has a fixed meaning.

4 Example

$p(m, n) = m < n$

TODO: look at different possibilities of $\forall \wedge \exists$.

5 Formal Definition of an n-place predicate

<u>Definition</u>: Let $\{U_i : i \in \{1, 2, \cdots n\}$ and $U_i \neq \emptyset\}$ be n universes. An n-place predicate is a function that maps $U_1 \times U_2 \times \cdots \times U_n$ to a set of propositions. That is an n-place predicate takes n free variables and produces a proposition based on the values of those variables. We can write a predicate as $p(x_1, x_2, \cdots, x_n)$ where x_i can vary over U_i.

If we bind one of the variables, say x_j then we get a new predicate $p(x_1, x_2, \cdots, a, \cdots, x_n)$ where a is either a bound variable of a $\forall$ or a $\exists$.

We will only get a true or false answer reliably if we bind all n variables.

6 example

$p(m, n) = n > 2^m$.

1. $\forall m \exists n p(m, n)$ 2. $\exists m \forall n p(m, n)$

7 Compound Predicates

We can defined compound predicates as follows:

1. All variables are compound predicates
2. All n-place predicates are compound predicates
3. if P and Q are compound predicates, the so are $P \wedge Q, P \vee Q, \neg P, P \rightarrow Q, P \leftrightarrow Q$.
4. If $P(x, \cdots)$ is a compound predicate with free variable x, then $\forall x P(x, \cdots)$ and $\exists x P(x, \cdots)$ are compound predicates with bound variable x.

We call a compound predicate with no free variables a compound proposition.

Example:

$\exists x \exists z p(x, z) \rightarrow \forall y \neg r(y)$