Section 13.3 - Infinite Sets

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1 Sets of the Same Cardinality

When 2 sets are finite, it's easy to tell if they have the same – just count them. For infinite sets, this obviously doesn't work!

1.1 Definition

Definition: 2 sets S and T have the same cardinality if $\exists$ a one-to-one correspondence

$$f : S \to T.$$

2 Countability

2.1 Definitions

2.1.1 Countably infinite

Definition: A set is countably infinite if has the same cardinality as $\mathbb{N}$.

2.1.2 Countable

Definition: A set is countable if it is finite or countably infinite. Intuitively, a set is countable if you can at least start enumerating its elements.

2.1.3 Uncountable

Definition: A set that is not countable is called uncountable.

3 Proving That Some Sets are Countable

Theorem 1. $|\mathbb{P}| = |\mathbb{N}|$

Proof. The one-to-one correspondence $f(p) = p - 1$ maps $\mathbb{P}$ onto $\mathbb{N}$ and is one to one. □

Theorem 2. $|\mathbb{E} = \{n \in \mathbb{N} : n = 2m \textit{ for some } m \in \mathbb{N}\}| = |\mathbb{N}|$

Proof. The one-to-one correspondence $f(n) = 2n$ maps $\mathbb{N}$ onto $\mathbb{E}$ and is one-to-one. □

Theorem 3. $|\mathbb{Z}| = |\mathbb{P}|$

Proof. We can write $\mathbb{Z}$ as $\{0, 1, -1, 2, -2, 3, -3, \cdots\}$. For any $x \in \mathbb{Z}$, we can find its corresponding element of $\mathbb{P}$ by the formula:

$$f(x) = \begin{cases} 2x + 1 & x \geq 0 \\ -2x & x < 0 \end{cases}$$

□

4 Uncountable sets

Theorem 4. $(0, 1)$ *is not countable.*

Proof. Suppose that there existed a one-to-correspondence between $(0, 1)$ and $\mathbb{P}$. Then we could list all of the elements of $(0, 1)$ in a table like this:

$$\begin{array}{r|l} 1 & 0.1110101... \\ 2 & 0.1231231... \\ 3 & 0.1451271... \\ 4 & 0.7981234... \\ 5 & 0.9867125... \end{array}$$

Now look at the diagonal. I can construct a new number as follows: If the digit is 0, change it to a 1 and if it's not a 0, change it to a 7 (or anything else, honestly.) This new number is not on the list. Therefore, there is no one-to-one correspondence. Consequently, $(0, 1)$ is not countable.

$\square$

Theorem 5 (There are some things money can't buy...)**.** *There are more decision problems than there are computer programs to solve them.*

Proof. A computer program is ultimately a string of zeros and ones. This string of zeros and ones can be converted into a number in $\mathbb{P}$ A decision problem can be viewed as a function $f : \mathbb{N}^k \to \{0, 1\}$ that is, a decision problem is a function that takes in some parameters as input and returns yes or no. This can be viewed as a string of 0's and 1's that can be converted into a number in $(0, 1)$. So the set of programs is countable and the set of problems is uncountable.

$\square$