Section 2.1 - Introduction to Logic

Ari Mermelstein

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1 Propositions

1.1 Definition

Definition: A proposition is a statement that is either true or false, but not both. That is, for a proposition P, we can assign to P the truth value "true" or "false."

1.2 examples

1. Julius Caesar was once president of the USA.
2. $2 + 2 = 4$.
3. The number 4 is positive and the number 3 is negative.
4. $x + y = y + x \quad \forall x, y \in \mathbb{R}$.
5. $2n = n^2$ for some $n \in \mathbb{N}$.
6. If the world is flat, then the USMNT is atop the Coca-Cola FIFA World Ranking list.

1.3 Beware!

Not all sentences are propositions. Examples:

1. Study for your exams!
2. Do your homework!

2 Logical connectives

2.1 Combining propositions

Consider two propositions p and q. Then,

1. $\neg p =$ "not p"
2. $p \wedge q =$ "p and q" $=$ the conjunction of p and q.
3. $p \vee q =$ "p or q" $=$ the disjunction of p and q. NOTE: This includes the possibility that p and q are both true.
4. $p \rightarrow q =$ "p implies q" $=$ "if p is true, then q is true."
5. $p \leftrightarrow q =$ "p if and only if q."

2.2 Examples

- f = the Earth is flat.
- l = I won the lottery
- m = "2+2 = 4"
- r = "It's raining outside."
- w = "There is water outside."

Write the following in logical notation:

1. The Earth is flat or I won the lottery.
2. The Earth is not flat.
3. If it's raining outside, then there is water outside.
4. I won the lottery if and only if 2+2 = 4.
5. It's not the case that "I won the lottery" and "the Earth is flat" are both true.

3 Converse, inverse, contrapositive

Consider two propositions p and q and the implication $p \to q$. Then,
Definition: The converse of $p \to q$ is $q \to p$.
Definition: The inverse of $p \to q$ is $\neg p \to \neg q$.
Definition: The contrapositive of $p \to q$ is $\neg q \to \neg p$. That is to say, the contrapositive is the converse of the inverse or the inverse of the converse.

Theorem 1 (Contrapositive theorem)**.** *The contrapositive of an implication always has the same truth value of the original implication. i.e.* $p \to q \iff \neg q \to \neg p$.

Proof. Stay tuned! □

4 Predicates

4.1 Definition

Definition: A predicate is a function that takes an element of a set U (the set under consideration) and produces an output in the set $\{0, 1\}$ (that is {true, false}.

4.1.1 Examples

1. p(n) = "n is even, $n \in \mathbb{Z}$."
2. p(S) = "$|\mathcal{P}(S)| = 2^{|S|}$ where S is a set."
3. p(n) = "$n^2 + 2n + 2$ is not a perfect square, $n \in \mathbb{N}$."

4.2 Quantifiers

There are 2 quantifiers: The existential quantifier and the universal quantifier. These allow us to build compound propositions.

4.2.1 Universal Quantifier

The universal quantifier, $\forall$, allows us to build compound propositions in the form

$$\forall x, p(x)$$

which means "for all x, p(x) is true" or "for every" or "for each."

The a compound proposition that uses the universal quantifier is only true if p(x) is true for all $x \in U$.

4.2.2 Existential Quantifier

The existential quantifer, $\exists$, allows us the build compound propositions of the form

$$\exists x, p(x)$$

which means "there exists an x such that p(x) is true." or "for some x, p(x) is true."

5 Disproving Compound Propositions

Suppose we have a compound proposition that uses the universal quantifier. What is necessary to prove a statement like that false? Counterexamples! A **counterexample** is a example for which the proposition is not true.

5.1 Examples

Provide a counterexample for the following:

1. $\forall n \in \mathbb{N}$, if x is prime, then x is also odd.
2. $\forall n \in \mathbb{N}, \quad n^2 \leq 2^n$.
3. $\forall n \in \mathbb{N}, \quad 2^n + 3^n$ is a prime number.