Section 2.2 - Propositional Calculus

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1 Truth Tables

1.1 Definition

Definition: A truth table is a table that lists all possible truth values for all variables in the proposition and shows the truth value of the proposition under those inputs.

2 Definitions for all Primitive Operations

2.1 not

p	¬ p
0	1
1	0

2.2 or

p	q	$p \lor q$
0	0	0
0	1	1
1	0	1
1	1	1

2.3 XOR

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

2.4 and

p	q	$p \land q$
0	0	0
0	1	0
1	0	0
1	1	1

2.5 implies

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

2.6 bi-conditional

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

3 Truth tables for compound propositions

3.1 Example 1

Build a truth table for $(p \wedge q) \vee \neg(p \to q)$.

p	q	$p \wedge q$	$p \to q$	$\neg(p \to q)$	$(p \wedge q) \vee \neg(p \to q)$
0	0	0	1	0	0
0	1	0	1	0	0
1	0	0	0	1	1
1	1	1	1	0	1

3.2 Example 2

Build a truth table for $(p \to q) \wedge [(q \wedge \neg r) \to (p \vee r)]$.

p	q	r	$p \to q$	$\neg r$	$q \wedge \neg r$	$p \vee r$	$(q \wedge \neg r) \to (p \vee r)$	$(p \to q) \wedge [(q \wedge \neg r) \to (p \vee r)]$
0	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	1
0	1	0	1	1	1	0	0	0
0	1	1	1	0	0	1	1	1
1	0	0	0	1	0	1	1	0
1	0	1	0	0	0	1	1	0
1	1	0	1	1	1	1	1	1
1	1	1	1	0	0	1	1	1

4 Tautologies and Contradictions

4.1 Definitions

Definition: A tautology is a compound proposition that is always true regardless of the values of the variables.
Definition: A contradiction is a compound proposition that is always false regardless of the values of the variables.

4.2 Examples

:

Theorem 1 (modus ponens). *The formula $[p \wedge (p \to q)] \to q$ is a tautology*

Proof.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Since this formula always has the value 1 for each possibility, the formula is a tautology. □

5 Logical Equivalence

Definition: 2 compound propositions P and Q are called logically equivalent if $P \leftrightarrow Q$ is a tautology. We write $P \iff Q$.

Theorem 2 (contrapositive rule)**.** $(p \to q) \iff (\neg q \to \neg p)$.

Proof.

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Since the two propositions always have the same value, they are logically equivalent. □

6 Logical Implication

Definition: We say that a compound proposition P **logically implies** Q if $P \to Q$ is a tautology. We write $P \implies Q$.

7 Some True Facts

1. $\neg\neg p \iff p$ double negation
2. $p \wedge q \iff q \wedge p$
   $p \vee q \iff q \vee p$ commutativity
3. $p \wedge (q \wedge r) \iff (p \wedge q) \wedge r$
   $p \vee (q \vee r) \iff (p \vee q) \vee r$ associativity
4. $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$ distributive laws
5. $\neg(p \wedge q) \iff \neg p \vee \neg q$
   $\neg(p \vee q) \iff \neg p \wedge \neg q$ De Morgan's laws.
6. $p \wedge (p \to q) \implies q$ modus ponens
7. $p \implies p \vee q$
8. $p \wedge q \implies p$ (also implies q obviously...)