Section 2.4 - Equivalence Relations

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May 8, 2018

1 Direct Proofs

Definition: A direct proof is a proof where you use the antecedent of the implication that you are trying to prove along with other theorems and facts in in order to prove a result.

Example:

Theorem 1. *If n is an even integer, then n^2 is also an even integer.*

Proof. Since n is even, we can write $n = 2k$ for some $k \in \mathbb{Z}$. Then, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is itself an integer, we conclude that n^2 is even. □

Theorem 2. *The product of an even number and an odd number is even.*

Proof. Let x be an even number and let y be an odd number. Then we can write $x = 2k$ and $y = 2m + 1$ for some $k, m \in \mathbb{Z}$. Then, $xy = 2k(2m + 1) = 4km + 2k = 2(2km + k)$. Since $2km + k$ is an integer, we conclude that xy is even. □

Sometimes we have to break up a problem into cases to accomplish a direct proof.

Theorem 3. *$n^2 - 2$ is not divisible by 3 for all $n \geq 1$.*

Proof. For any number n, the remainder when you divide it by 3 is either 0,1, or 2. Therefore, we can write any n as $n = 3k+r$ where $r \in \{0, 1, 2\}$. So $n^2 - 2 = (3k+r)^2 - 2 = 9k^2 + 6rk + r^2 - 2$. We see that the $9k^2 + 6rk$ part is divisible by 3. What about $r^2 - 2$? Well, it either equals $0 - 2 = -2$ or $1 - 2 = -1$ or $4 - 2 = 2$, none of which are divisible by 3. □

2 Proof by contrapositive

Sometimes its easier to prove the contrapositive of an implication than the implication itself.

Theorem 4. *If n^2 is an even integer, then n is also an even integer.*

Proof. Since n^2 is hard to quantify, let's prove the contrapositive. Namely, let's prove that if n is NOT even, then n^2 is also NOT even.

If n is odd, we can write $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ which is an odd number. □

3 Proof by Contradiction

Theorem 5. *If $\sqrt{2}$ is not a rational number.*

Proof. Suppose for purposes of contradiction that $\sqrt{2} = \dfrac{a}{b}$ where a and b have been reduced to lowest terms. Then $2 = \dfrac{a^2}{b^2} \implies 2b^2 = a^2$. This tells us that a^2 is even. By the Theorem 4 above, a is also even. Therefore, let $a = 2k$. We can rewrite $a^2 = (2k)^2 = 4k^2 = 2b^2$. This implies that b is also even. If a and b are both even, then they weren't really reduced to lowest terms. Contradiction! □

Theorem 6. *There are (countably) infinitely many primes.*

Proof. Suppose that there were only finitely many primes. Then, you could list all of the prime numbers $p_1, p_2, \cdots, p_n$. Consider $k = p_1 p_2 p_3 \cdots p_n + 1$. This number is not divisible by any of $p_1, p_2, \cdots p_n$. Therefore, either this number is a prime number itself, or it has a prime factor that isn't on the list. Contradiction! $\square$