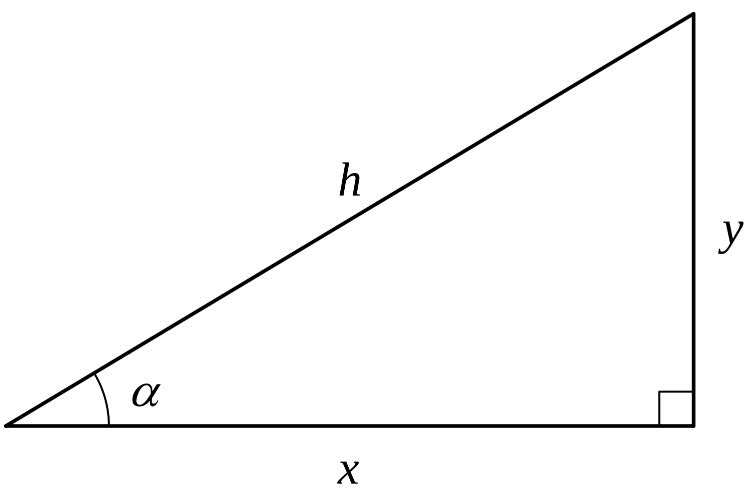
# Chapter 4.1
# Mathematical Concepts

# Applied Trigonometry

- "Old Henry And His Old Aunt"
  - Defined using right triangle




$$\sin\alpha = \frac{y}{h}$$

$$\cos\alpha = \frac{x}{h}$$

$$\tan\alpha = \frac{y}{x} = \frac{\sin\alpha}{\cos\alpha}$$

# Applied Trigonometry

- Angles measured in radians

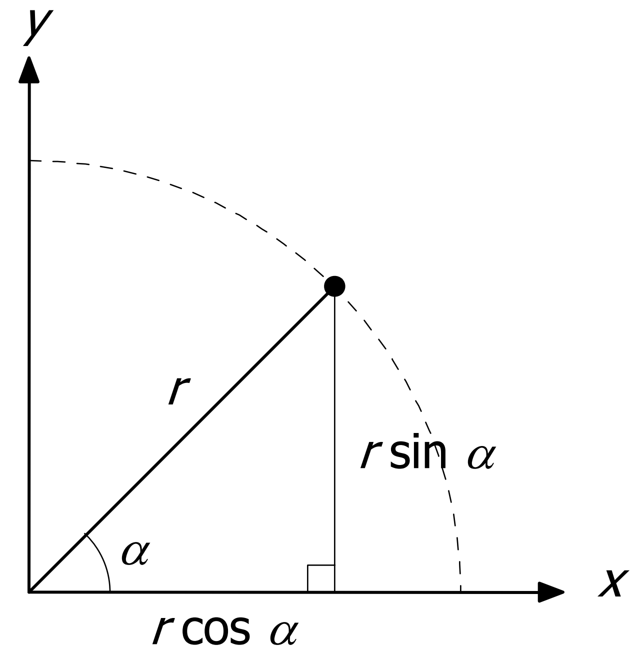
$$radians = \frac{\pi}{180}(degrees)$$

$$degrees = \frac{180}{\pi}(radians)$$

- Full circle contains $2\pi$ radians

# Applied Trigonometry

- Sine and cosine used to decompose a point into horizontal and vertical components



# Applied Trigonometry

- Trigonometric identities

$$\sin(-\alpha) = -\sin\alpha$$
$$\cos(-\alpha) = \cos\alpha$$
$$\tan(-\alpha) = -\tan\alpha$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos\alpha = \sin(\alpha + \pi/2)$$
$$\sin\alpha = \cos(\alpha - \pi/2)$$

$$\cos\alpha = -\sin(\alpha - \pi/2)$$
$$\sin\alpha = -\cos(\alpha + \pi/2)$$

$$\sin\alpha = -\sin(\alpha + \pi) = -\sin(\alpha - \pi)$$
$$\cos\alpha = -\cos(\alpha + \pi) = -\cos(\alpha - \pi)$$

# Applied Trigonometry

- Inverse trigonometric (arc) functions
  - Return angle for which sin, cos, or tan function produces a particular value
  - If $\sin \alpha = z$, then $\alpha = \sin^{-1} z$
  - If $\cos \alpha = z$, then $\alpha = \cos^{-1} z$
  - If $\tan \alpha = z$, then $\alpha = \tan^{-1} z$

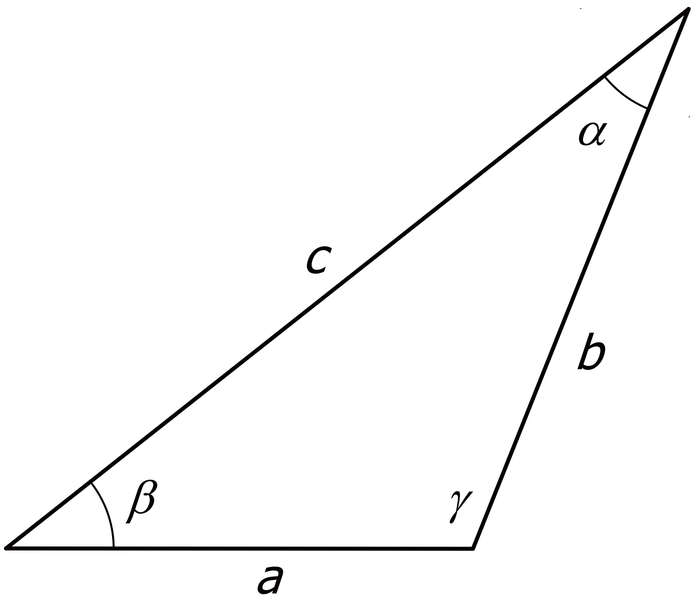
# Applied Trigonometry

- Law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

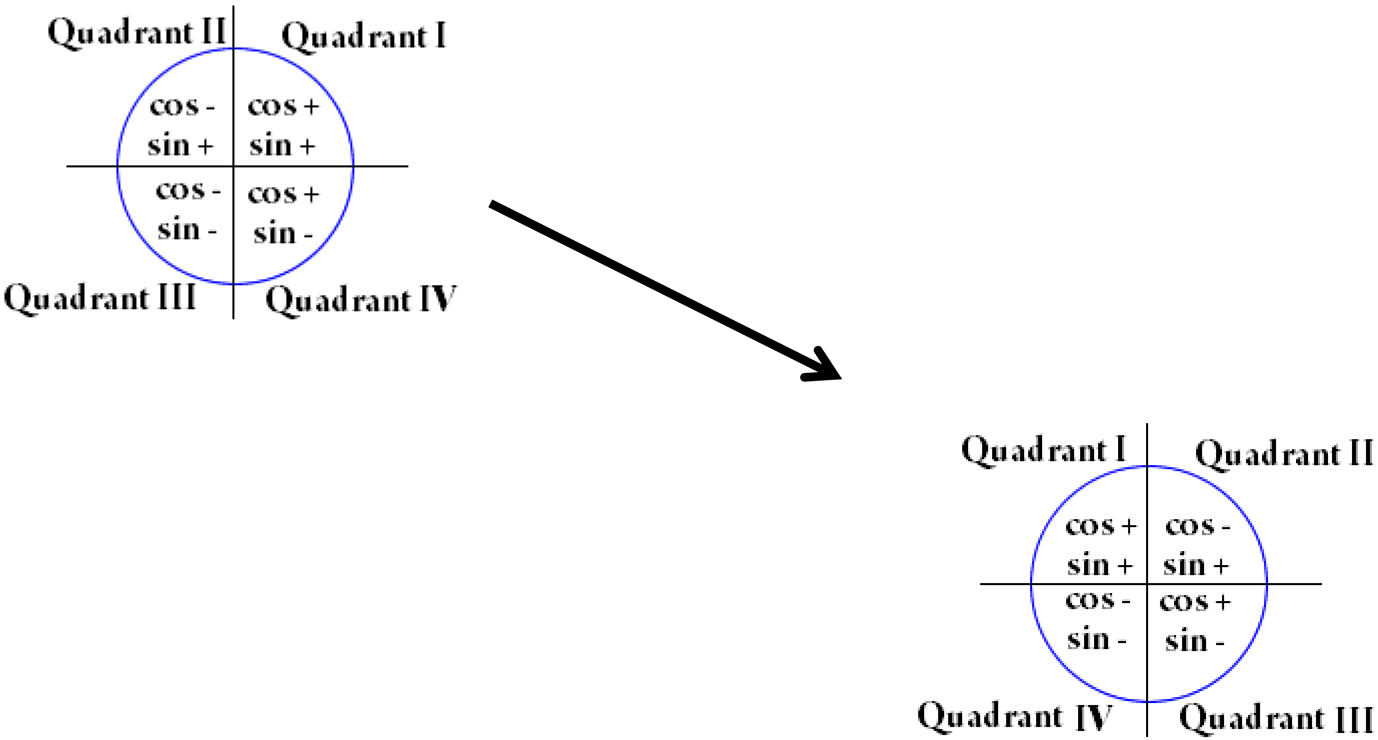
- Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$




  - Reduces to Pythagorean theorem when $\gamma$ = 90 degrees

# Trigonometric Identities



# Scalars & Vectors & Matrices (oh my!)

# Scalars & Vectors

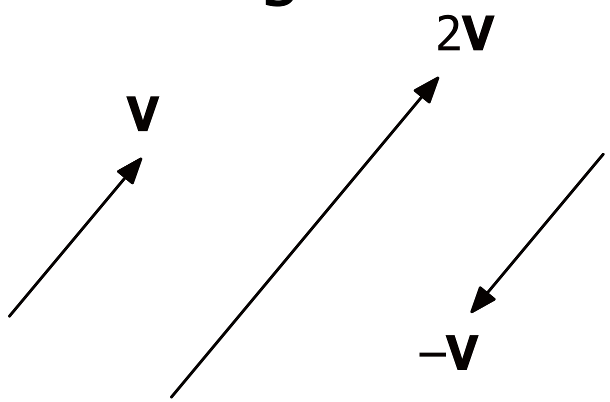
- Scalars represent quantities that can be described fully using one value
  - Mass
  - Time
  - Distance
- Vectors describe a 'state' using multiple values (magnitude and direction together)

# Vectors

- Examples of vectors
  - Difference between two points
    - Magnitude is the distance between the points
    - Direction points from one point to the other
  - Velocity of a projectile
    - Magnitude is the speed of the projectile
    - Direction is the direction in which it's traveling
  - A force is applied along a direction

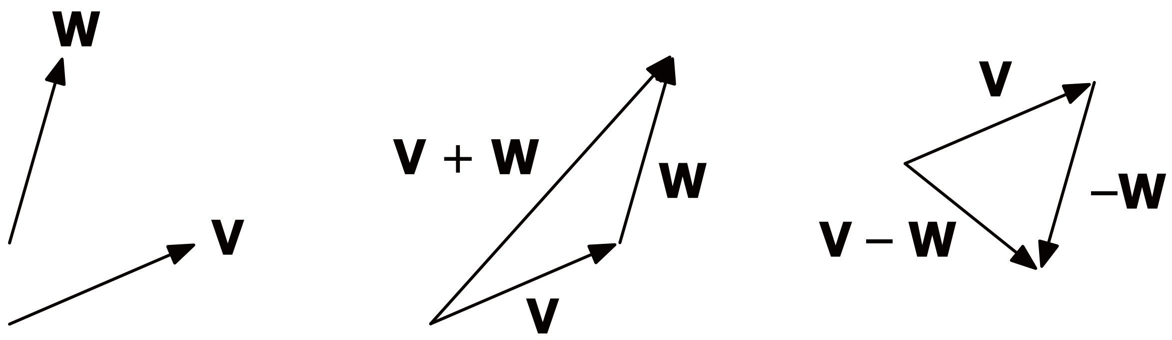
# Vectors (cont)

- Vectors can be visualized by an arrow
  - The length represents the magnitude
  - The arrowhead indicates the direction
  - Multiplying a vector by a scalar changes the arrow's length



# Vectors Mathematics

- Two vectors **V** and **W** are added by placing the beginning of **W** at the end of **V**
- Subtraction reverses the second vector



# 3D Vectors

- An *n*-dimensional vector **V** is represented by *n* components
- In three dimensions, the components are named *x*, *y*, and *z*
- Individual components are expressed using the name as a subscript:

$$\mathbf{V} = \langle 1, 2, 3 \rangle \qquad V_x = 1 \qquad V_y = 2 \qquad V_z = 3$$

# Vector Mathematics

- Vectors add and subtract componentwise

$$\mathbf{V}+\mathbf{W}=\langle V_1+W_1, V_2+W_2, \ldots, V_n+W_n\rangle$$
$$\mathbf{V}-\mathbf{W}=\langle V_1-W_1, V_2-W_2, \ldots, V_n-W_n\rangle$$

# Magnitude of a Vector

- The magnitude of an *n*-dimensional vector **V** is given by

$$\|\mathbf{V}\| = \sqrt{\sum_{i=1}^{n} V_i^2}$$

- In three dimensions (Pythagoras 3D)

$$\|\mathbf{V}\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

- Distance from the origin.

# Normalized Vectors

- A vector having a magnitude of 1 is called a unit vector
- Any vector **V** can be resized to unit length by dividing it by its magnitude:

$$\hat{\mathbf{V}} = \frac{\mathbf{V}}{\|\mathbf{V}\|}$$

- This process is called normalization
- Piecewise division

# Matrices

- A matrix is a rectangular array of numbers arranged as rows and columns
  - A matrix having $n$ rows and $m$ columns is an $n \times m$ matrix
  - At the right, **M** is a $2 \times 3$ matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

  - If $n = m$, the matrix is a square matrix

# Matrices

- The entry of a matrix **M** in the *i*-th row and *j*-th column is denoted $M_{ij}$
- For example,

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$M_{11} = 1 \quad M_{21} = 4$$

$$M_{12} = 2 \quad M_{22} = 5$$

$$M_{13} = 3 \quad M_{23} = 6$$

# Matrices Transposition

- The transpose of a matrix **M** is denoted $\mathbf{M}^T$ and has its rows and columns exchanged:

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{M}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

# Vectors and Matrices

- An $n$-dimensional vector **V** can be thought of as an $n \times 1$ column matrix:

$$\mathbf{V} = \langle V_1, V_2, \ldots, V_n \rangle = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

- Or a $1 \times n$ row matrix:

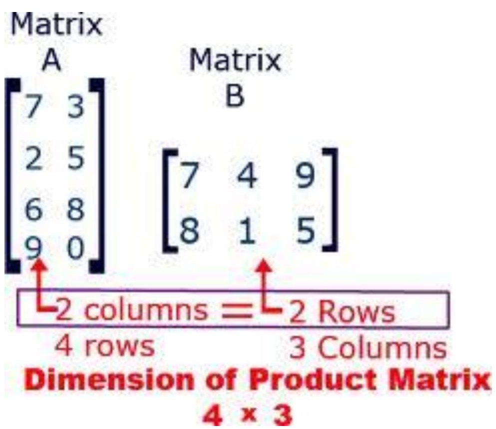
$$\mathbf{V}^{\mathrm{T}} = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}$$

# Matrix Multiplication

- Product of two matrices **A** and **B**
  - Number of columns of **A** must equal number of rows of **B**
  - If **A** is a $n \times m$ matrix, and **B** is an $m \times p$ matrix, then **AB** is an $n \times p$ matrix
  - Entries of the product are given by

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

# Example




$$x = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} * \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$$

$$1*9+2*8+3*7=46$$

$$4*9+5*8+6*7=118$$

$$x = \begin{pmatrix} 46 \\ 118 \end{pmatrix}$$

# More Examples

- Example matrix product

$$\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 8 & -13 \\ -6 & 6 \end{bmatrix}$$

$$\begin{aligned} M_{11} &= 2\cdot(-2)+3\cdot 4 &&= 8 \\ M_{12} &= 2\cdot 1+3\cdot(-5) &&= -13 \\ M_{21} &= 1\cdot(-2)+(-1)\cdot 4 &&= -6 \\ M_{22} &= 1\cdot 1+(-1)\cdot(-5) &&= 6 \end{aligned}$$

# Coordinate Systems (more later)

- Matrices are used to transform vectors from one coordinate system to another
- In three dimensions, the product of a matrix and a column vector looks like:

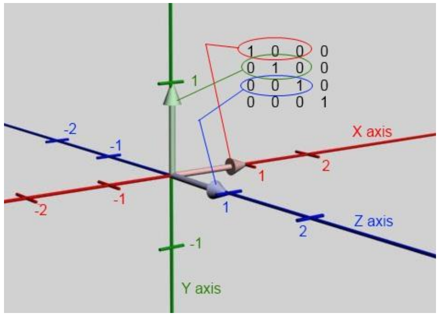
$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} M_{11}V_x + M_{12}V_y + M_{13}V_z \\ M_{21}V_x + M_{22}V_y + M_{23}V_z \\ M_{31}V_x + M_{32}V_y + M_{33}V_z \end{bmatrix}$$

# Identity Matrix

- An $n \times n$ identity matrix is denoted $\mathbf{I}_n$
- $\mathbf{I}_n$ has entries of 1 along the main diagonal and 0 everywhere else

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

# Identity Matrix




- For any $n \times n$ matrix $\mathbf{M}_n$, the product with the identity matrix is $\mathbf{M}_n$ itself
  - $\mathbf{I}_n \mathbf{M}_n = \mathbf{M}_n$
  - $\mathbf{M}_n \mathbf{I}_n = \mathbf{M}_n$
- The identity matrix is the matrix analog of the number one.

# Inverse & Invertible

- An $n \times n$ matrix **M** is invertible if there exists another matrix **G** such that

$$\mathbf{MG} = \mathbf{GM} = \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- The inverse of **M** is denoted $\mathbf{M}^{-1}$

# Determinant

- **Not every matrix has an inverse!**
- A noninvertible matrix is called singular.
- Whether a matrix is invertible or not can be determined by calculating a scalar quantity called the **determinant.**

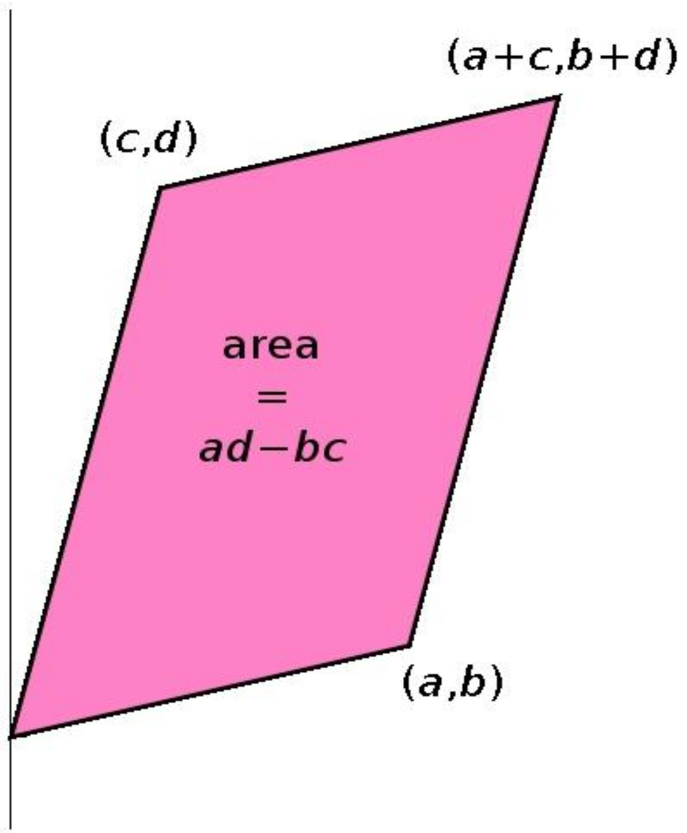
# Determinant

- The determinant of a square matrix **M** is denoted det **M** or |**M**|
- A matrix is invertible if its determinant <u>is not zero</u>
- For a 2 $\times$ 2 matrix,

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

# 2D Determinant

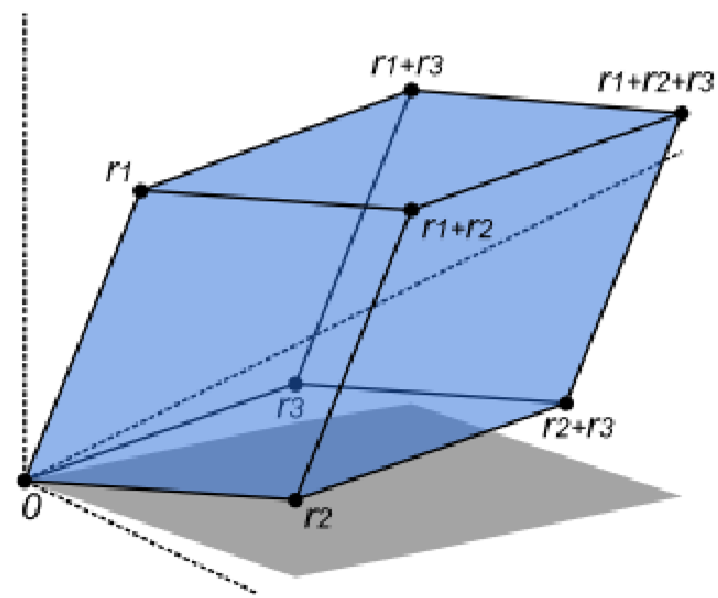
- Can also be thought as the are of a parallelogram



# 3D Determinant

det (A) = aei + bfg + cdh − afh − bdi − ceg.

$$
\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}
\quad - \quad
\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}
$$



# Calculating matrix inverses

- If you have the determinant you can find the inverse of a matrix.
- A decent tutorial can be found here:
- http://easycalculation.com/matrix/inverse-matrix-tutorial.php
- For the most part you will use a function to do the busy work for you.

# Officially "New" Stuff

# The Dot Product

- The dot product is a product between two vectors that produces a scalar
  - The dot product between two *n*-dimensional vectors **V** and **W** is given by

$$\mathbf{V} \cdot \mathbf{W} = \sum_{i=1}^{n} V_i W_i$$

  - In three dimensions,

$$\mathbf{V} \cdot \mathbf{W} = V_x W_x + V_y W_y + V_z W_z$$

# The Dot Product

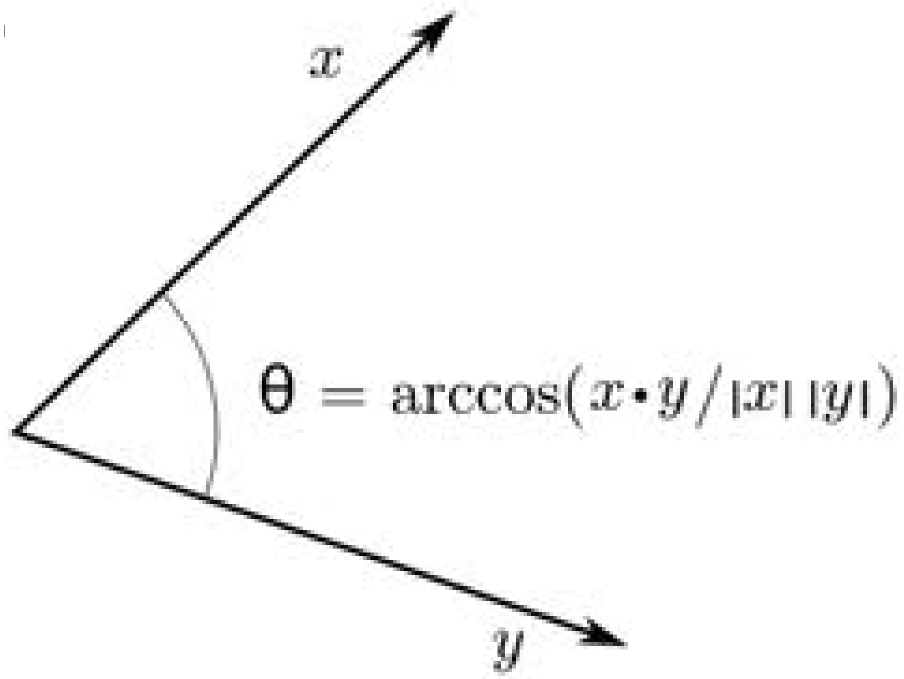
- The dot product satisfies the formula

$$\mathbf{V} \cdot \mathbf{W} = \|\mathbf{V}\|\|\mathbf{W}\| \cos \alpha$$

  - $\alpha$ is the angle between the two vectors
  - ||V|| magnitude.
  - Dot product is always 0 between perpendicular vectors (Cos 90 = 0)
  - If **V** and **W** are unit vectors, the dot product is 1 for parallel vectors pointing in the same direction, -1 for opposite

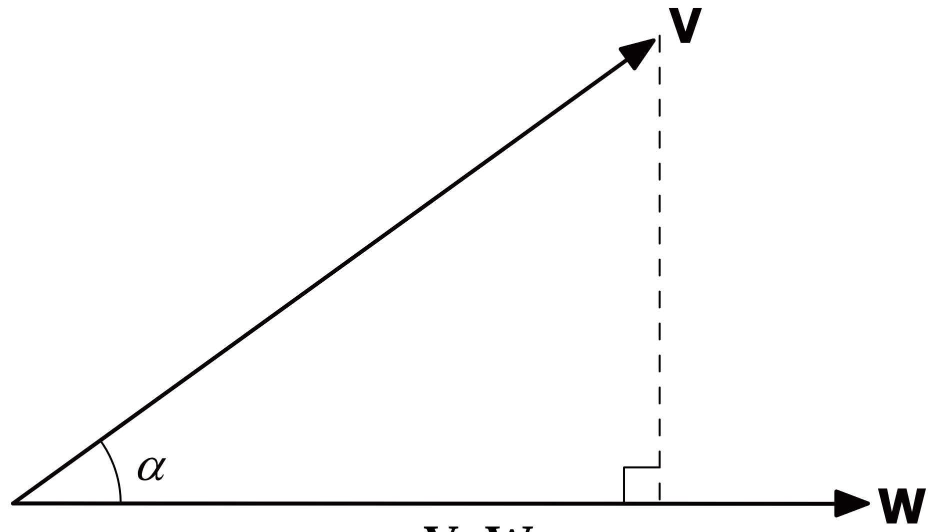
# Dot Product

- Solving the previous formula for Θ yields



# The Dot Product

- The dot product can be used to project one vector onto another




$$\|\mathbf{V}\|\cos\alpha = \frac{\mathbf{V}\cdot\mathbf{W}}{\|\mathbf{W}\|}$$

# The Dot Product

- The dot product of a vector with itself produces the squared magnitude

$$\mathbf{V} \cdot \mathbf{V} = \|\mathbf{V}\|\|\mathbf{V}\| = \|\mathbf{V}\|^2$$

- Often, the notation $V^2$ is used as shorthand for $\mathbf{V} \cdot \mathbf{V}$

# Dot Product Review

- Takes two vectors and makes a scalar.
  - Determine if two vectors are perpendicular
  - Determine if two vectors are parallel
  - Determine angle between two vectors
  - Project one vector onto another
  - Determine if vectors on same side of plane
  - Determine if two vectors intersect (as well as the when and where).
  - Easy way to get squared magnitude.

Whew….

# The Cross Product

- The cross product is a product between two vectors the produces a vector
  - The cross product **<u>only applies in three dimensions</u>**
  - The cross product of two vectors, is another vector, that has the property of being perpendicular to the vectors being multiplied together
  - The cross product between two parallel vectors is the zero vector (0, 0, 0)

# The Cross Product

- The cross product between **V** and **W** is

$$\mathbf{V}\times\mathbf{W}=\left\langle V_yW_z-V_zW_y,\, V_zW_x-V_xW_z,\, V_xW_y-V_yW_x\right\rangle$$

- A helpful tool for remembering this formula is the pseudodeterminant

$$\mathbf{V}\times\mathbf{W}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ V_x & V_y & V_z\\ W_x & W_y & W_z\end{vmatrix}$$

# The Cross Product

- The cross product can also be expressed as the matrix-vector product

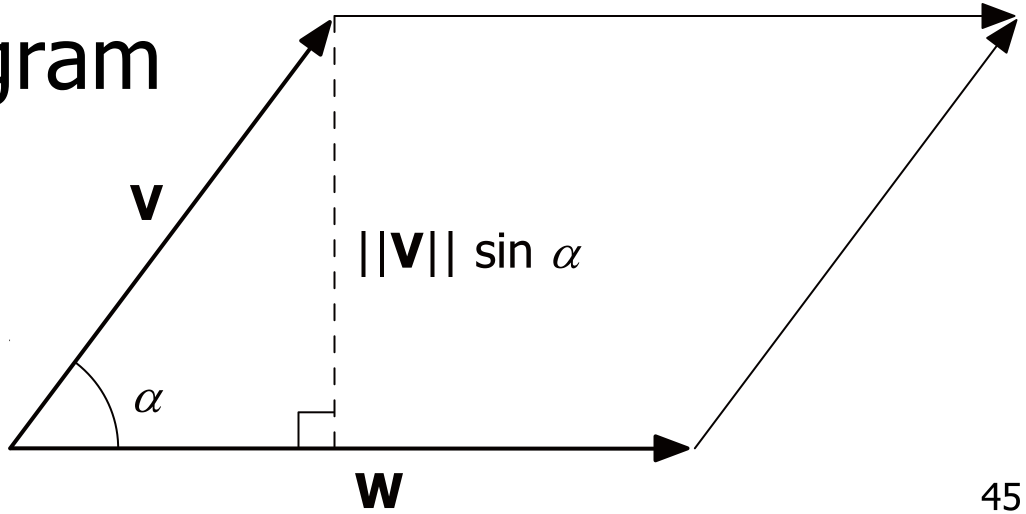
$$\mathbf{V} \times \mathbf{W} = \begin{bmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{bmatrix} \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix}$$

# The Cross Product

- The cross product satisfies the trigonometric relationship

$$\|\mathbf{V} \times \mathbf{W}\| = \|\mathbf{V}\| \|\mathbf{W}\| \sin \alpha$$

- This is the area of the parallelogram formed by **V** and **W**



# The Cross Product

- The area $A$ of a triangle with vertices $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ is thus given by

$$A = \frac{1}{2} \left\| \left( \mathbf{P}_2 - \mathbf{P}_1 \right) \times \left( \mathbf{P}_3 - \mathbf{P}_1 \right) \right\|$$

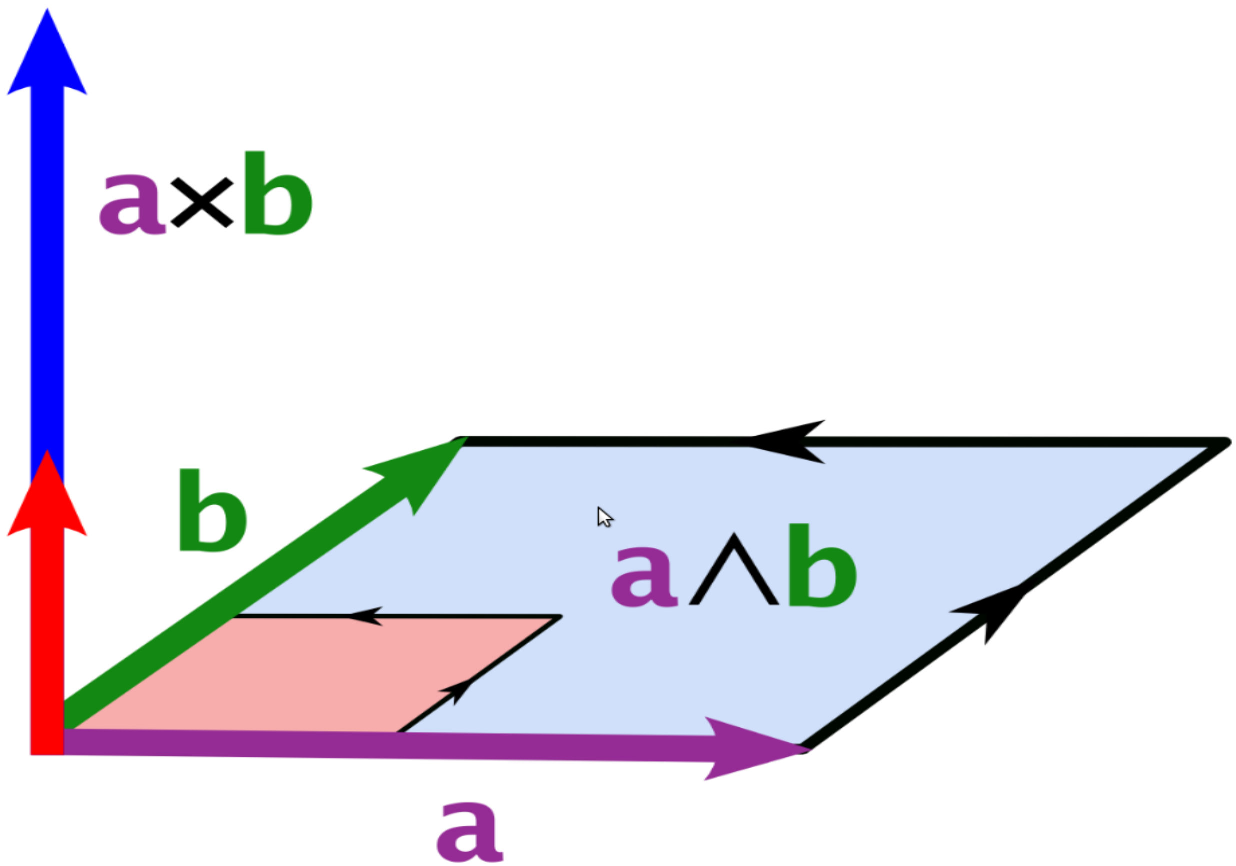
# The Cross Product

- Cross products obey the right hand rule
  - If first vector points along right thumb, and second vector points along right fingers,
  - Then cross product points out of right palm
- Reversing order of vectors negates the cross product:

$$\mathbf{W} \times \mathbf{V} = -\mathbf{V} \times \mathbf{W}$$

  - Cross product is anticommutative

# Cross Product Review



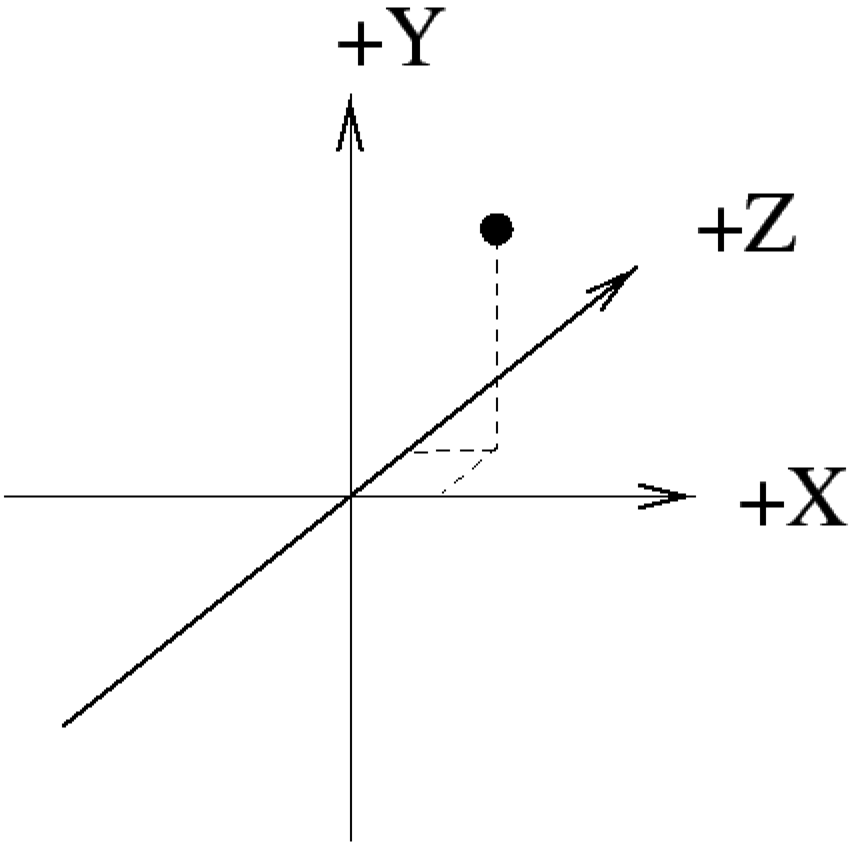
Almost there.... Almost there.

# Transformations

- Calculations are often carried out in many different coordinate systems
- We must be able to transform information from one coordinate system to another easily
- Matrix multiplication allows us to do this

# Transform Simplest Case

- Simplest case is inverting one or more axis.



# Transformations

- Suppose that the coordinate axes in one coordinate system correspond to the directions **R**, **S**, and **T** in another
- Then we transform a vector **V** to the **RST** system as follows

$$\mathbf{W} = [\mathbf{R} \quad \mathbf{S} \quad \mathbf{T}]\mathbf{V} = \begin{bmatrix} R_x & S_x & T_x \\ R_y & S_y & T_y \\ R_z & S_z & T_z \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

# Transformations

- We transform back to the original system by inverting the matrix:

$$\mathbf{V} = \begin{bmatrix} R_x & S_x & T_x \\ R_y & S_y & T_y \\ R_z & S_z & T_z \end{bmatrix}^{-1} \mathbf{W}$$

- Often, the matrix's inverse is equal to its transpose—such a matrix is called orthogonal

# Transformations

- A 3 × 3 matrix can reorient the coordinate axes in any way, but it leaves the origin fixed
- We must at a translation component **D** to move the origin:

$$\mathbf{W} = \begin{bmatrix} R_x & S_x & T_x \\ R_y & S_y & T_y \\ R_z & S_z & T_z \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} + \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

# Transformations

- Homogeneous coordinates
  - Four-dimensional space
  - Combines 3 × 3 matrix and translation into one 4 × 4 matrix

$$\mathbf{W} = \begin{bmatrix} R_x & S_x & T_x & D_x \\ R_y & S_y & T_y & D_y \\ R_z & S_z & T_z & D_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \\ V_w \end{bmatrix}$$

# Transformations

- **V** is now a four-dimensional vector
  - The $w$-coordinate of **V** determines whether **V** is a point or a direction vector
  - If $w = 0$, then **V** is a direction vector and the fourth column of the transformation matrix has no effect
  - If $w \neq 0$, then **V** is a point and the fourth column of the matrix translates the origin
  - Normally, $w = 1$ for points

# Transformations

- Transformation matrices are often the result of combining several simple transformations
  - Translations
  - Scales
  - Rotations
- Transformations are combined by multiplying their matrices together

# Transformations

- Translation matrix

$$\mathbf{M}_{\text{translate}} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Translates the origin by the vector **T**

# Transformations

- Scale matrix

$$\mathbf{M}_{\text{scale}} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Scales coordinate axes by $a$, $b$, and $c$
- If $a = b = c$, the scale is uniform

# Transformations

- Rotation matrix

$$\mathbf{M}_{z\text{-rotate}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotates points about the *z*-axis through the angle $\theta$

# Transformations

- Similar matrices for rotations about *x*, *y*

$$\mathbf{M}_{x\text{-rotate}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{y\text{-rotate}} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Transformations Review

- We may wish to change an objects orientation, or it's vector information (Translate, Scale, Rotate, Skew).
- Storing an objects information in Vector form allows us to manipulate it in many ways at once.
- We perform those manipulations using matrix multiplication operations.

# Transforms in Flash

- http://help.adobe.com/en_US/ActionScript/3.0_ProgrammingAS3/WSF24A5A75-38D6-4a44-BDC6-927A2B123E90.html
- private var rect2:Shape;

- var matrix:Matrix3D = rect2.transform.matrix3D;
- matrix.appendRotation(15, Vector3D.X_AXIS);
- matrix.appendScale(1.2, 1, 1);
- matrix.appendTranslation(100, 50, 0);
- matrix.appendRotation(10, Vector3D.Z_AXIS);
- rect2.transform.matrix3D = matrix;

# Great Tutorials

- **2D Transformation**
- Rotating, Scaling and Translating

- **3D Transformation**
- Defining a Point Class
- 3d Transformations Using Matrices
- Projection

- **Vectors in Flash CS4**
- Adobe Library Link (note Dot and Cross Product)