MAT 2440 - HW3 Solutions

October 2016

Section 2.1

Exercise 20

a: 0
b: 1
c: 2
d: 3

Exercise 32

a

$$A \times B \times C = \{(a,x,0),(a,x,1),(a,y,0),(a,y,1),(b,x,0),(b,x,1),(b,y,0),(b,y,1),(c,x,0),(c,x,1),(c,y,0),(c,y,1)\}$$

d

$$B \times B \times B = \{(x,x,x),(x,x,y),(x,y,x),(y,x,x),(y,y,x),(y,x,y),(x,y,y),(y,y,y)\}$$

Section 2.2

Exercise 16

a

If $x \in A \cap B$ then $x \in A \wedge x \in B \Rightarrow x \in A$.
So we conclude that if $x \in A \cap B$ then $x \in A$ which means that $(A \cap B) \subseteq A$.

b

If $x \in A$ then $x \in A \vee x \in B \Rightarrow x \in A \cup B$.
So we conclude that if $x \in A$ then $x \in A \cup B$ which means that $A \subseteq (A \cup B)$.

c

If $x \in A - B$ then $x \in A \land x \notin B \Rightarrow x \in A$.
Therefore $(A - B) \subseteq A$.

d

Suppose that there exists an x such that $x \in A \cap (B - A)$. In this case $x \in A \land (x \in B \land x \notin A) \Rightarrow x \in A \land x \notin A$ which is a contradiction. Therefore, there exists no such element, and we have that $A \cap (B - A) = \emptyset$.

Exercise 49

a

We have that for all $i \leq j$ then $A_i \subseteq A_j$ because we can see that if a bit string has length of at most i then its length will surely be less than j.
This means that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$. Finally this means that $\bigcup_{i=1}^{n} A_i = A_n$.

b

Following the same reasoning as in *(a)* we have that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$. This means that $\bigcap_{i=1}^{n} A_i = A_1$.

Section 2.3

Exercise 12

a

Yes.
If we choose 2 elements $n_1, n_2 \in \mathbb{Z}$ such that $n_1 \neq n_2$ then if $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1 \Rightarrow n_1 = n_2$.

b

No.
If we choose 2 elements $n_1, n_2 \in \mathbb{Z}$ such that $n_1 \neq n_2$ then if $f(n_1) = f(n_2) \Rightarrow {n_1}^2 + 1 = {n_2}^2 + 1 \Rightarrow {n_1}^2 = {n_2}^2 \Rightarrow n_1 = \pm n_2$.
Alternatively, we can pick a non-zero n_1 and $-n_1$ and it's obvious that $f(n_1) = f(-n_1)$ although clearly $n_1 \neq -n_1$.

c

Yes.
If we choose 2 elements $n_1, n_2 \in \mathbb{Z}$ such that $n_1 \neq n_2$ then if $f(n_1) = f(n_2) \Rightarrow {n_1}^3 = {n_2}^3 \Rightarrow n_1 = n_2$.

Exercise 14

a

Yes.
We can produce every integer using the expression $2m - n$.

b

No.
As a counterexample we can see that there is no (m, n) that can give us the integer $k = 4$.

c

Yes.
We can produce every integer using the expression $m + n + 1$.

Exercise 20

a

Let $f : \mathbb{N} \to \mathbb{N}$ and $f(x) = x^3$.
Now, f is one-to-one (as we saw before) but the natural number $n = 4$ is not the image of any x. So f is not onto $\mathbb{N}$.

b

Let $f : \mathbb{N} \to \mathbb{N}$ and $f(x) = \lfloor \frac{x}{2} \rfloor$.
Now, f is not one-to-one because for every positive even integer n, n and $n+1$ are both always sent to the same number. But f is onto $\mathbb{N}$ because, for every integer n in the codomain, the integer $2n$ in the domain is sent to n under f.

c

If we define a function $f : \mathbb{N} \to \mathbb{N}$ as:

$$f(x) = \begin{cases} x+1 & x \text{ is even} \\ x-1 & x \text{ is odd} \end{cases}$$

then we map even numbers (counting 0 as even), and odd numbers to even numbers. This function is one-to-one and onto.

d

If we define a function $f : \mathbb{N} \to \mathbb{N}$ as:

$$f(x) = \begin{cases} 1 & x \text{ is even} \\ 0 & x \text{ is odd} \end{cases}$$

then this function is neither one-to-one nor onto.

Exercise 58

a

Since 1 byte = 8 bits, 1 byte is enough to encode 4 bits.

b

Since 1 byte = 8 bits, and 2 bytes = 16 bits, we need 2 byte is enough to encode 10 bits.

Remark

What we actually do, is use the ceiling function. The number b of bytes needed to encode n bits is $b = \lceil \frac{n}{8} \rceil$.

Section 2.4

Exercise 32

a

$$\sum_{j=0}^{8} 1 + (-1)^j = \sum_{j=0}^{8} 1 + \sum_{j=0}^{8} (-1)^j = 9 + 1 = 10$$

b

$$\sum_{j=0}^{8} 3^j - 2^j = \sum_{j=0}^{8} 3^j - \sum_{j=0}^{8} 2^j = 1 \cdot \frac{3^9 - 1}{3 - 1} - 1 \cdot \frac{2^9 - 1}{2 - 1} = 9330$$

c

$$\sum_{j=0}^{8} 2 \cdot 3^j + 3 \cdot 2^j = 2 \cdot \sum_{j=0}^{8} 3^j + 3 \cdot \sum_{j=0}^{8} 2^j = 2 \cdot \frac{3^9 - 1}{3 - 1} + 3 \cdot \frac{2^9 - 1}{2 - 1} = 21215$$

d

$$\sum_{j=0}^{8} 2^{j+1} - 2^j = \sum_{j=0}^{8} 2^j \cdot (2 - 1) = \sum_{j=0}^{8} 2^j = 1 \cdot \frac{2^9 - 1}{2 - 1} = 511$$