Enhanced 2-adjunctions

New York City Category Theory Seminar
City University of New York

Igor Baković

Josip Juraj Strossmeyer University of Osijek, Croatia

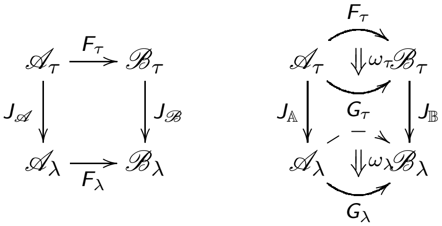
February 2, 2023

Table of contest

Formal properties of enhanced categories

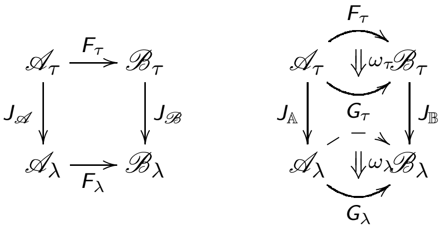
The category of enhanced categories

- We start with the category $\mathscr{F}$ whose objects are injective on objects, fully faithful functors



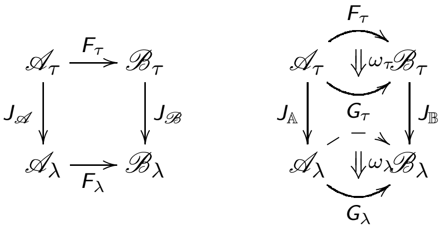
The category of enhanced categories

- We start with the category $\mathscr{F}$ whose objects are injective on objects, fully faithful functors
- It is a cartesian closed category which Lack and Shulman used in 2012 as a basis of enrichment



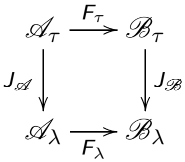
The category of enhanced categories

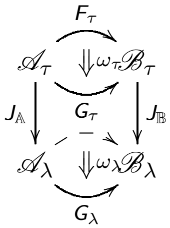
- We start with the category $\mathscr{F}$ whose objects are injective on objects, fully faithful functors
- It is a cartesian closed category which Lack and Shulman used in 2012 as a basis of enrichment
- The objects of $\mathscr{F}$-$\mathscr{C}at$ are called *enhanced 2-categories*



The category of enhanced categories

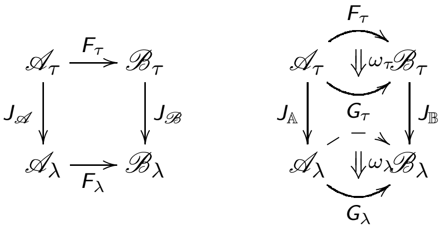
- We start with the category $\mathscr{F}$ whose objects are injective on objects, fully faithful functors
- It is a cartesian closed category which Lack and Shulman used in 2012 as a basis of enrichment
- The objects of $\mathscr{F}$-$\mathscr{C}at$ are called *enhanced 2-categories*
- These are precisely bijective on objects, faithful and locally fully faithful 2-functors





The category of enhanced categories

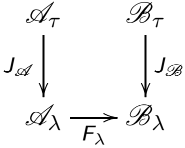
- We start with the category $\mathscr{F}$ whose objects are injective on objects, fully faithful functors
- It is a cartesian closed category which Lack and Shulman used in 2012 as a basis of enrichment
- The objects of $\mathscr{F}$-$\mathscr{C}at$ are called *enhanced 2-categories*
- These are precisely bijective on objects, faithful and locally fully faithful 2-functors
- 1-cells of $\mathscr{F}$ are commuting squares and 2-cells are cylinders



- What are nice properties of $\mathscr{F}$ investigated by Lack and Shulman?

$$\begin{array}{ccc} \mathscr{A}_\tau & & \mathscr{B}_\tau \\ {\scriptstyle J_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathscr{B}}} \\ \mathscr{A}_\lambda & \xrightarrow[F_\lambda]{} & \mathscr{B}_\lambda \end{array}$$

- What are nice properties of $\mathscr{F}$ investigated by Lack and Shulman?
- Full embeddings are the right class of a factorization system on $\mathscr{C}at$ (the left class consists of functors that are surjective on objects).



- What are nice properties of $\mathscr{F}$ investigated by Lack and Shulman?
- Full embeddings are the right class of a factorization system on $\mathscr{C}at$ (the left class consists of functors that are surjective on objects).
- $\mathscr{F}$ is reflective in $\mathscr{C}at^2$ and therefore complete and cocomplete, with limits formed pointwise.

$$\begin{array}{ccc} \mathscr{A}_\tau & & \mathscr{B}_\tau \\ {\scriptstyle J_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathscr{B}}} \\ \mathscr{A}_\lambda & \xrightarrow[F_\lambda]{} & \mathscr{B}_\lambda \end{array}$$

- What are nice properties of $\mathscr{F}$ investigated by Lack and Shulman?
- Full embeddings are the right class of a factorization system on $\mathscr{C}at$ (the left class consists of functors that are surjective on objects).
- $\mathscr{F}$ is reflective in $\mathscr{C}at^2$ and therefore complete and cocomplete, with limits formed pointwise.
- Colimits in $\mathscr{F}$ are formed by taking the colimit in $\mathscr{C}at^2$, then applying the reflection, which amounts to taking the full embedding part of the (surjective on objects, full embedding) factorization of a functor.

$$\begin{array}{ccc} \mathscr{A}_\tau & & \mathscr{B}_\tau \\ {\scriptstyle J_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathscr{B}}} \\ \mathscr{A}_\lambda & \xrightarrow[F_\lambda]{} & \mathscr{B}_\lambda \end{array}$$

- What are nice properties of $\mathscr{F}$ investigated by Lack and Shulman?
- Full embeddings are the right class of a factorization system on $\mathscr{C}at$ (the left class consists of functors that are surjective on objects).
- $\mathscr{F}$ is reflective in $\mathscr{C}at^2$ and therefore complete and cocomplete, with limits formed pointwise.
- Colimits in $\mathscr{F}$ are formed by taking the colimit in $\mathscr{C}at^2$, then applying the reflection, which amounts to taking the full embedding part of the (surjective on objects, full embedding) factorization of a functor.
- $\mathscr{F}$ itself can be enriched to an enhanced 2-category whose loose part consists of loose 1-cells

$$\begin{array}{ccc} \mathscr{A}_\tau & & \mathscr{B}_\tau \\ {\scriptstyle J_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathscr{B}}} \\ \mathscr{A}_\lambda & \xrightarrow[F_\lambda]{} & \mathscr{B}_\lambda \end{array}$$

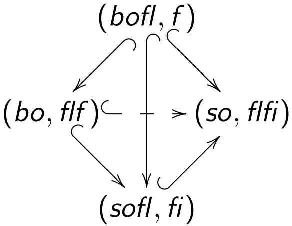
- What additional structure there is in and around $\mathscr{F}$ which was not investigated by Lack and Shulman..?

- What additional structure there is in and around $\mathscr{F}$ which was not investigated by Lack and Shulman..?
- There is a ternary factorization system on $\mathscr{C}at$ determined by the inclusion $(sofl, fi) \subset (so, flfi)$ of two (strict) 2-categorical factorization systems where $(sofl, fi)$ stands for (surjective on objects and full, injective on objects and faithful) and $(so, flfi)$ for (surjective on objects, injective on objects and fully faithful).This ternary factorization system induces a diagram

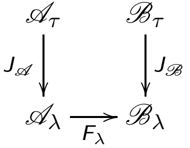
$$\begin{array}{ccc} \mathscr{C}at(sofl, fi) & \longrightarrow & \mathscr{C}at(so, flfi) \\ & {\scriptstyle Ob}\searrow \quad \swarrow{\scriptstyle Ob} & \\ & \mathscr{S}et(sur, inj) & \end{array} \tag{1.1}$$

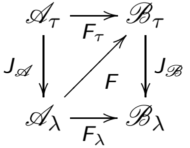
Every ternary factorization system $(L_1, R_1) \subset (L_2, R_2)$ generates three classes of maps (E, F, M) where $E = L_1$, $F = L_2 \cap R_1$ and $M = R_2$.

In our case three classes (*sofl*, *bof*, *flfi*) are surjective on objects and full,bijective on objects and faithful, fully faithful inclusions. Note that the last class of maps are precisely the objects of $\mathscr{F}$! This ternary factorization system involves the generalization of Postnikov tower theory - (n-connected, n-truncated) factorization system - from groupoids to categories (directed homotopy types). The ternary factorization system $(sofl, fi) \subset (so, flfi)$ extends to a quaternary factorization system $(bofl, f) \subset (sofl, fi) \subset (so, flfi)$ where $(bofl, f)$ is (bijective on objects and full, faithful). There is even quinary factorization system



- Lack and Shulman showed how $\mathscr{F}$ itself can be enriched to an enhanced 2-category or $\mathscr{F}$-category whose loose part $\mathscr{F}_\lambda$ consists of loose 1-cells





The class of tight functors is right ideal with respect to loose functors!

- Lack and Shulman showed how $\mathscr{F}$ itself can be enriched to an enhanced 2-category or $\mathscr{F}$-category whose loose part $\mathscr{F}_\lambda$ consists of loose 1-cells

$$\begin{array}{ccc} \mathscr{A}_\tau & & \mathscr{B}_\tau \\ {\scriptstyle J_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathscr{B}}} \\ \mathscr{A}_\lambda & \xrightarrow[F_\lambda]{} & \mathscr{B}_\lambda \end{array}$$

- However, there exist a third class of *tight* 1-cells between enhanced categories

$$\begin{array}{ccc} \mathscr{A}_\tau & \xrightarrow[F_\tau]{} & \mathscr{B}_\tau \\ {\scriptstyle J_{\mathscr{A}}}\downarrow & \overset{F}{\nearrow} & \downarrow{\scriptstyle J_{\mathscr{B}}} \\ \mathscr{A}_\lambda & \xrightarrow[F_\lambda]{} & \mathscr{B}_\lambda \end{array}$$

The class of tight functors is right ideal with respect to loose functors!

[redacted]

There exists a closed structure on the category $\mathscr{F}$ whose internal hom is a 2-functor

$$[-,-]\colon \mathscr{F}^{op} \times \mathscr{F} \to \mathscr{F}$$

which sends any pair of enhanced categories $\mathscr{A}$ and $\mathscr{B}$ to an enhanced category $[\mathscr{A},\mathscr{B}]$ of loose functors whose tight objects are tight functors. The unit for this closed structure is a unique functor $1_\lambda\colon \emptyset \to *$ from the empty category to the terminal category. Furthermore, the above functor extends to a 2-functor

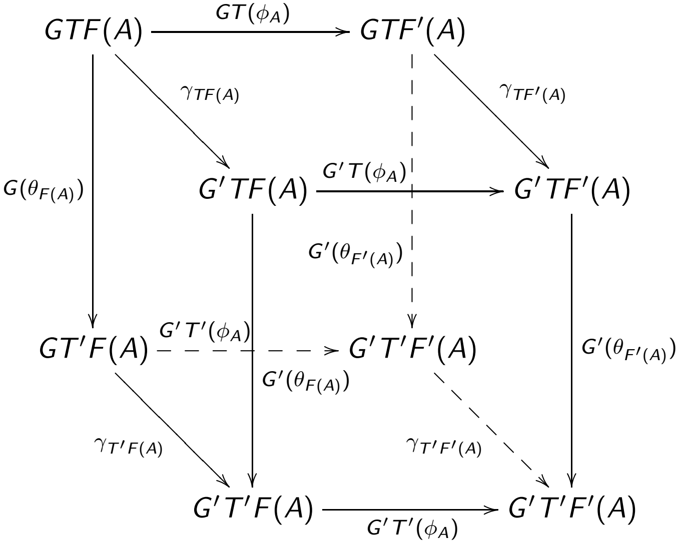
$$[-,-]\colon \mathscr{F}_\lambda^{op} \times \mathscr{F}_\lambda \to \mathscr{F}_\lambda$$

which sends any pair $F_\lambda\colon \mathscr{A}'_\lambda \to \mathscr{A}_\lambda$ and $G_\lambda\colon \mathscr{B}_\lambda \to \mathscr{B}'_\lambda$ of loose functors to a loose functor $[F_\lambda, G]\colon [\mathbb{A},\mathbb{B}] \to [\mathbb{A}',\mathbb{B}']$ taking any loose $T_\lambda\colon \mathscr{A}_\lambda \to \mathscr{B}_\lambda$ or tight functor $T\colon \mathscr{A}_\lambda \to \mathscr{B}_\tau$ to $G_\lambda T_\lambda F_\lambda$ and $G_\lambda T F_\lambda$ respectively.

The action of internal homon any two 2-cells $\phi\colon F' \Rightarrow F$ and $\gamma\colon G \Rightarrow G'$ in $\mathbb{F}$ is for any loose $T_\lambda\colon \mathscr{A}_\lambda \to \mathscr{B}_\lambda$ or tight functor $T\colon \mathscr{A}_\lambda \to \mathscr{B}_\tau$ given by a commutative square of 2-cells

$$\begin{array}{ccc} [F,G](T) & \overset{[\phi,\gamma]_T}{\Longrightarrow} & [F',G'](T) \\ {\scriptstyle [F,G](\theta)}\Downarrow & & \Downarrow{\scriptstyle [F',G'](\theta)} \\ [F,G](T') & \underset{[\phi,\gamma]_{T'}}{\Longrightarrow} & [F',G'](T') \end{array}$$

in $\mathscr{F}$, where $[\phi,\gamma]_T\colon [F,G](T) \Rightarrow [F',G'](T)$ is an (enhanced) transformation whose component indexed by any object A in $\mathbb{A}'$ is defined by a diagonal of the top square in the following cube


$GTF(A)$
$GT(\phi_A)$
$GTF'(A)$
$\gamma_{TF(A)}$
$\gamma_{TF'(A)}$
$G(\theta_{F(A)})$
$G'TF(A)$
$G'T(\phi_A)$
$G'TF'(A)$
$G'(\theta_{F'(A)})$
$GT'F(A)$
$G'T'(\phi_A)$
$G'T'F'(A)$
$G'(\theta_{F'(A)})$
$G'(\theta_{F(A)})$
$\gamma_{T'F(A)}$
$\gamma_{T'F'(A)}$
$G'T'F(A)$
$G'T'F'(A)$
$G'T'(\phi_A)$

[redacted]

A category $\mathscr{A}$ enriched in a closed structure on the category $\mathscr{F}$ with respect to the internal hom $[-,-]\colon \mathscr{F}^{op} \times \mathscr{F} \to \mathscr{F}$ is a 2-category with a right ideal of 1-cells.

A category $\mathscr{A}$ enriched in $\mathscr{F}$ consists of a class of objects $\mathscr{A}_0$ together with an object $\mathscr{A}(A, B)$ of $\mathscr{F}$ for any two objects A and B in $\mathscr{A}_0$. We call tight objects of $\mathscr{A}(A, B)$ admissible 1-cells of $\mathscr{A}$. For each object A in $\mathscr{A}_0$ there is an enhanced functor

$$J_A\colon 1_\lambda \to \mathscr{A}(A, A)$$

which lifts to a unit 1_τ of the cartesian cosed structure of $\mathscr{F}$ if and only if A is admissible object. Finally,for any three objects A, B and C the enhanced functor

$$L^A_{B,C}\colon \mathscr{A}(B, C) \to [\mathscr{A}(A, B), \mathscr{A}(A, C)]$$

is defined by postcomposition.

- I was able to define (co)lax functors in the enriched context $\mathscr{F}_{[-,-]}$-$\mathscr{C}at$ (probably the most complicated coherence diagram in my life represented by Stasheff assochiahedron) together with (co)lax transformations and modifications.

- I was able to define (co)lax functors in the enriched context $\mathscr{F}_{[-,-]}$-$\mathscr{C}at$ (probably the most complicated coherence diagram in my life represented by Stasheff assochiahedron) together with (co)lax transformations and modifications.
- For any two $\mathscr{F}_{[-,-]}$ enriched categories $\mathscr{K}$ and Ł, there exist a $\mathscr{F}_{[-,-]}$-category $[\mathscr{K},\mathcal{L}]$ whose admissible 1-cells which form a right ideal are enriched transformations with components indexed by objects admissible 1-cells

- I was able to define (co)lax functors in the enriched context $\mathscr{F}_{[-,-]}$-$\mathscr{C}at$ (probably the most complicated coherence diagram in my life represented by Stasheff assochiahedron) together with (co)lax transformations and modifications.
- For any two $\mathscr{F}_{[-,-]}$ enriched categories $\mathscr{K}$ and Ł, there exist a $\mathscr{F}_{[-,-]}$-category $[\mathscr{K}, \mathcal{L}]$ whose admissible 1-cells which form a right ideal are enriched transformations with components indexed by objects admissible 1-cells
- These are internal homs for an enriched functor

$$[-,-]\colon \mathscr{F}_{[-,-]} - \mathscr{C}at^{op} \times \mathscr{F}_{[-,-]} - \mathscr{C}at \to \mathscr{F}_{[-,-]} - \mathscr{C}at$$

which makes a class of 2-categories with right ideals closed

- I was able to define (co)lax functors in the enriched context $\mathscr{F}_{[-,-]}$-$\mathscr{C}at$ (probably the most complicated coherence diagram in my life represented by Stasheff assochiahedron) together with (co)lax transformations and modifications.
- For any two $\mathscr{F}_{[-,-]}$ enriched categories $\mathscr{K}$ and Ł, there exist a $\mathscr{F}_{[-,-]}$-category $[\mathscr{K}, \mathcal{L}]$ whose admissible 1-cells which form a right ideal are enriched transformations with components indexed by objects admissible 1-cells
- These are internal homs for an enriched functor

$$[-,-]\colon \mathscr{F}_{[-,-]}-\mathscr{C}at^{op} \times \mathscr{F}_{[-,-]}-\mathscr{C}at \to \mathscr{F}_{[-,-]}-\mathscr{C}at$$

  which makes a class of 2-categories with right ideals closed
- The theory of Yoneda structures by Street and Walters has a natural and beautiful description in this enriched context.

- I was able to define (co)lax functors in the enriched context $\mathscr{F}_{[-,-]}$-$\mathscr{C}at$ (probably the most complicated coherence diagram in my life represented by Stasheff assochiahedron) together with (co)lax transformations and modifications.
- For any two $\mathscr{F}_{[-,-]}$ enriched categories $\mathscr{K}$ and Ł, there exist a $\mathscr{F}_{[-,-]}$-category $[\mathscr{K},\mathcal{L}]$ whose admissible 1-cells which form a right ideal are enriched transformations with components indexed by objects admissible 1-cells
- These are internal homs for an enriched functor

$$[-,-]\colon \mathscr{F}_{[-,-]}-\mathscr{C}at^{op}\times \mathscr{F}_{[-,-]}-\mathscr{C}at \to \mathscr{F}_{[-,-]}-\mathscr{C}at$$

  which makes a class of 2-categories with right ideals closed
- The theory of Yoneda structures by Street and Walters has a natural and beautiful description in this enriched context.
- The underlying structure is described by enhanced modules.

The second closed structure on $\mathscr{F}$ is defined by a functor

$$\langle -,-\rangle \colon \mathscr{F}^{op} \times \mathscr{F} \to \mathscr{F}$$

which takes any pair of enhanced categories $\mathbb{A}$ and $\mathbb{B}$ to an enhanced category $\langle \mathbb{A}, \mathbb{B}\rangle$ whose tight and loose objects are tight and enhanced functors respectively. For any two enhanced functors $F \colon \mathbb{A}' \to \mathbb{A}$ and $G \colon \mathbb{B} \to \mathbb{B}'$ loose and tight components of the enhanced functor $\langle F, G\rangle \colon \langle \mathbb{A}, \mathbb{B}\rangle \to \langle \mathbb{A}', \mathbb{B}'\rangle$ are defined by $\langle F, G\rangle(T) := GTF$ for any enhanced functor $T \colon \mathbb{A} \to \mathbb{B}$ and $\langle F, G\rangle(T)$ is tight if T is. However, (**??**) does not extend to an enhanced functor $\langle -,-\rangle \colon \mathbb{F}^{op} \times \mathbb{F} \to \mathbb{F}$ but it does extend to an enhanced 2-functor

$$\langle -,-\rangle \colon \mathbb{F}_\tau^{op} \times \mathbb{F}_\tau \to \mathbb{F}_\tau$$

where $\mathbb{F}_\tau$ is an enhanced 2-category whose loose part is $\mathscr{F}$ and whose tight 1-cells are tight functors. Categories enriched with respect to this closed structure are categories with a 2-sided ideal.

[redacted]

An enhanced bicategory $\mathscr{B}$ consists of a homomorphism of bicategories

$$\begin{array}{c} \mathscr{B}_\tau \\ \Big\downarrow{\scriptstyle J_{\mathscr{B}}} \\ \mathscr{B}_\lambda \end{array}$$

bijective on objects, faithful and locally fully faithful. We say that $\mathscr{B}_\tau$ is the tight part and $\mathscr{B}_\lambda$ is the loose part of $\mathscr{B}$, and we say that (**??**) is strongly enhanced bicategory if its tight part is a 2-category.

[redacted]

For any two enhanced bicategories $\mathscr{A}$ and $\mathscr{B}$ a loose morphism $F_\lambda \colon \mathscr{A}_\lambda \to \mathscr{B}_\lambda$ is a (co)morphism between the loose parts. A loose morphism is enhanced if there exists a (necessarily unique) homomorphism $F_\tau \colon \mathscr{A}_\tau \to \mathscr{B}_\tau$ such that

$$\begin{array}{ccc} \mathscr{A}_\tau & \xrightarrow{F_\tau} & \mathscr{B}_\tau \\ {\scriptstyle J_{\mathbb{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathbb{B}}} \\ \mathscr{A}_\lambda & \xrightarrow[F_\lambda]{} & \mathscr{B}_\lambda \end{array}$$

is a commutative diagram in the category *Bicat* of bicategories and their morphisms and

$$\varphi^F_{g,f} \colon F_\lambda(g) \circ F_\lambda(f) \Rightarrow F_\lambda(g \circ f)$$

[redacted]

There exists enhanced 2-categories $\mathscr{S}\mathscr{F}\mathrm{Bicat}$, $\mathscr{S}\mathscr{F}\mathrm{Bicat}_{co}$, $\mathscr{S}\mathscr{F}\mathrm{Bicat}_{op}$ and $\mathscr{S}\mathscr{F}\mathrm{Bicat}_{coop}$ which all have strongly enhanced bicategories as objects. Their loose 1-cells are enhanced morphisms, comorphisms, opmorphimsms and coopmorphisms respectively, and they all have enhanced homomorphisms as tight 1-cells. The class of 2-cells are enhanced transformations in the first two cases and enhanced optransformations in the last two. They are sub-2-categories of corresponding tricategories $\mathscr{F}\mathrm{Bicat}$, $\mathscr{F}\mathrm{Bicat}_{co}$, $\mathscr{F}\mathrm{Bicat}_{op}$ and $\mathscr{F}\mathrm{Bicat}_{coop}$ of (not necessarily strongly) enhanced bicategories, which have enhanced modifications as 3-cells. Strongly enhanced bicategories are also objects of the enhanced 2-categories $\mathscr{T}\mathscr{F}\mathrm{Bicat}_{w}$, where w stands for any of the above subscripts, with the same loose 1-cells and 2-cells and whose tight 1-cells are tight homomorphims.

[redacted]

The proof follows the same pattern as the one which describes the

[redacted]

The enhanced 2-category $\mathscr{S}\mathscr{F}\mathsf{Bicat}_{coop}$ *has an internal hom enhanced 2-functor*

$$[-,-]\colon \mathscr{F}\mathscr{C}\mathsf{at}^{\mathsf{op}} \times \mathscr{F}\mathscr{C}\mathsf{at} \to \mathscr{F}\mathscr{C}\mathsf{at}$$

which is on the level of objects defined for any two enhanced 2-categories $\mathscr{A}$ *and* $\mathscr{B}$ *as an enhanced 2-category* $[\mathscr{A},\mathscr{B}]$ *whose objects are enhanced 2-functors, tight and loose 1-cells are tight and enhanced optransformations and 2-cells are modifications.*

[redacted]

□

- R. Betti, A. J. Power, On local adjointness of distributive bicategories, Bollettino U.M.I. 7, (1988), 931-947.
- M. Bunge, Coherent Extensions and Relational Algebras, Transactions of the American Mathematical Society, Vol. 197 (1974), 355-390.
- C.B. Jay, Local adjunctions, Journal of Pure and Applied Algebra 53, (1988) 227-238.
- P.T. Johnstone, Fibrations and partial products in a 2-category, Applied Categorical Structures, Vol. 1, Nr. 2 (1993), 141-179.
- R. Street, Fibrations and Yoneda's lemma in a 2-category, Category Seminar (Proc. Sem., Sydney, 1972/1973), Lecture Notes in Mathematics, Vol. 420, (1974) 104-133.
- R. Street, R. Walters, Yoneda structures on 2-categories, Journal of Algebra Volume 50, Issue 2, February 1978, Pages 350-379