The Logical Theory of Canonical Maps:
The Elements & Distinctions Analysis of the Morphisms, Duality, Canonicity, and Universal Constructions in *Sets*

David Ellerman
University of Ljubljana, Slovenia

Abstract

Category theory gives a mathematical characterization of naturality but not of canonicity. The purpose of this paper is to develop the logical theory of canonical maps based on the broader demonstration that the dual notions of elements & distinctions are the basic analytical concepts needed to unpack and analyze morphisms, duality, canonicity, and universal constructions in *Sets*, the category of sets and functions. The analysis extends directly to other *Sets*-based concrete categories (groups, rings, vector spaces, etc.). Elements and distinctions are the building blocks of the two dual logics, the Boolean logic of subsets and the logic of partitions. The partial orders (inclusion and refinement) in the lattices for the dual logics define morphisms. The thesis is that the maps that are canonical in *Sets* are the ones that are defined (given the data of the situation) by these two logical partial orders and by the compositions of those maps.

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1 Elements & Distinctions Analysis

1.1 Introduction

Category theory gives a mathematical characterization of naturality but not of canonicity, the canonical nature of certain maps. The purpose of this paper is to present the *logical theory of canonical maps* that provides such a characterization. That logical theory of canonical maps is one of the main results in the broader analysis showing that the dual notions of "elements & distinctions" (or "its & dits") are the basic analytical concepts needed to unpack and analyze morphisms, duality, canonicity (or canonicalness), and universal constructions in the $Sets$, the category of sets and functions. The analysis extends directly to other $Sets$-based concrete categories (groups, rings, vector spaces, etc.) where the objects are sets with a certain type of structure and the morphisms are set functions that preserve or reflect that structure. Then the elements & distinctions-based definitions can be abstracted in purely arrow-theoretic way for abstract category theory.

One way to approach the concepts of "elements" (or "its") and "distinctions" (or "dits") is to start with the category-theoretic duality between subsets and quotient sets (= partitions = equivalence relations): "The dual notion (obtained by reversing the arrows) of 'part' [subobject] is the notion of *partition*." [8, p. 85]. That motivates the two dual forms of mathematical logic: the Boolean logic of subsets and the logic of partitions ([3]; [4]). If partitions are dual to subsets, then what is the dual concept that corresponds to the notion of *elements of a subset*? The notion dual to the elements of a subset is the notion of the *distinctions of a partition* (pairs of elements in distinct blocks of the partition).

1.2 The logical theory of *canonical* maps based on the its & dits analysis

Jean-Pierre Marquis [9] has raised the question of characterizing canonical maps in mathematics in general and category theory in particular. Category theory gives a mathematical notion of "naturality" but not of canonicalness or canonicity. Marquis gives the intuitive idea (maps defined "without any arbitrary decision"), a number of examples (most of which we will analyze in $Sets$), and a set of criteria stated in terms of limits (and thus dually for colimits).

> We are now in a position to circumscribe more precisely what we want to include in the notion of canonical morphisms or maps.
>
> 1. Morphisms that are part of the data of a limit are canonical morphisms; for instance, the projection morphisms that are part of the notion of a product;
>
> 2. The unique morphism from a cone to a limit determined by a universal property is a canonical morphism: and
>
> 3. In particular, the unique isomorphism that arise between two candidates for a limit is a canonical morphism. [9, p. 101]

The elements & distinctions (or its & dits) analysis provides a *mathematical* characterization of "canonical maps" in $Sets$ (and thus in $Sets$-based concrete categories) that satisfies the Marquis criteria.

The canonical maps and the unique canonical factor morphisms in the universal mapping properties in $Sets$ are always constructed in the two ways that maps are constructed from the logical partial orders in the two basic logics, the logic of subsets and the logic of partitions. In the powerset Boolean algebra of subsets $\wp(U)$ of U, the partial order is the inclusion relation $S \subseteq T$ for $S, T \subseteq U$, which induces the canonical injection $S \to T$. That is the way canonical injective maps are defined from the partial order of inclusion on subsets.

In the dual algebra of partitions $\Pi(U)$ on U (i.e., the lattice of partitions on U enriched with the implication operation on partitions[1]), the partial order is the refinement relation between partitions and it induces a canonical map using refinement. A *partition* $\pi = \{B, B', ...\}$ on a set U is a set of non-empty subsets of U (called *blocks*, $B, B', ...$) that are mutually exclusive (i.e., disjoint) and jointly exhaustive (i.e., whose union is U). It might be noticed that the empty set $\emptyset$, which has no nonempty subsets, is the empty partition on $U = \emptyset$.[2] One could also define a partition on U as the set of inverse-images $f^{-1}(y)$ for $y \in f(U)$ for any function $f : U \to Y$. The union of the inverse images is all of U since a function transmits elements and the inverse images are disjoint because a function reflects distinctions (see below for this analysis of a function).

Given another partition $\sigma = \{C, C', ...\}$ on U, a partition π is said to *refine* σ (or σ is *refined by* π), written $\sigma \precsim \pi$, if for every block $B \in \pi$, there is a block $C \in \sigma$ (necessarily unique) such that $B \subseteq C$. If we denote the set of distinctions or dits of a partition (ordered pairs of elements in different blocks) by $\operatorname{dit}(\pi)$, the ditset of π, then just as the partial order in $\wp(U)$ is the inclusion of elements, so the refinement partial order on $\Pi(U)$ is the inclusion of distinctions, i.e., $\sigma \precsim \pi$ iff (if and only if) $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$. And just as the inclusion ordering on subsets induces a canonical map between subsets, so the refinement ordering $\sigma \precsim \pi$ on partitions induces a canonical surjection between partitions, namely $\pi \to \sigma$ where $B \in \pi$ is taken to the unique $C \in \sigma$ where $B \subseteq C$. That is the way canonical surjective maps are defined from the partial order of refinement on partitions.

These canonical injections and surjections are *built into* the partial orders of the lattice (or algebraic) structure of the two dual logics of subsets and partitions; they logically define the 'atomic' *canonical* maps in *Sets*, and other canonical maps in *Sets* arise out of their compositions. This *logical theory of canonical maps* is that all "canonical" maps in *Sets* arise in these two ways or by compositions of them–which then extends to *Sets*-based concrete categories. The thesis cannot be proven since "canonical" is an intuitive notion. But we will show that all the canonical maps and unique factor maps in the universal constructions (limits and colimits) in *Sets* arise in this way from the partial orders of the dual lattices (or algebras) of subsets and partitions–which thus satisfies the Marquis criteria. This logical basis for this theory of canonical maps accounts for the name "logical." In *Sets*-based concrete categories, there may be maps that are part of the structure of the structured sets which can be taken as given 'by-definition' canonical maps in those categories and which may be composed with the canonical maps from the underlying category *Sets* (that have to arise in the specified ways) to make more canonical morphisms.

1.3 Initial & terminal objects and epi-mono factorization in *Sets*

The top of the powerset Boolean algebra $\wp(U)$ is U, where each subset $S \subseteq U$ induces the canonical injection $S \to U$. The bottom of the Boolean algebra, the null set $\emptyset$, is included in any set, e.g., $\emptyset \subseteq U$, so the induced morphism $\emptyset \to U$ is the canonical map that makes $\emptyset$ the initial object in *Sets* (taking U as any set).

The top of the partition algebra $\Pi(U)$ is the *discrete partition* $\mathbf{1}_U = \{\{u\}\}_{u \in U}$ of all singletons. Since every partition π is refined by $\mathbf{1}_U$, i.e., $\pi \precsim \mathbf{1}_U$, there is the canonical surjection $\mathbf{1}_U \cong U \to \pi$ that takes the singleton $\{u\}$ or just u (since blocks of $\mathbf{1}_U$ are in one-to-one correspondence with the elements of U) to the unique block B such that $u \in B$, i.e., to that point $B \in \pi$ when π is considered as a quotient set. The bottom of the partition algebra (or lattice) is the *indiscrete partition* (nicknamed the "blob") $\mathbf{0}_U = \{U\}$ with only one block U that identifies all the points in

[1] In [4], the partition algebra was defined as the partition lattice enriched with the implication and nand operations on partitions. But for purposes of comparisons with Boolean or Heyting algebras, it suffices to consider only the implication in addition to the join and meet. In any case, this does not affect the analysis here where the lattice structure suffices.

[2] Thanks to Paul Blain Levy and Alex Simpson for emphasizing to me the role of empty partition for the consistent development of the whole theory, e.g., as the inverse-image partition on the domain of the empty function $\emptyset \to Y$.

U so, as a quotient set, $\mathbf{0}_U$ is just the one-element set 1. And $\mathbf{0}_U$ is refined by all partitions, e.g., $\mathbf{0}_U \precsim \mathbf{1}_U$. That refinement relation induces the unique map from the blocks of $\mathbf{1}_U$ (i.e., the elements of U) to the blocks or rather the block of $\mathbf{0}_U$, i.e., induces the canonical map $U \to 1$ that makes 'the' one-element set 1 into the terminal object in *Sets* (taking U as any set).

Thus the maps induced by the top-bottom inclusion/refinement relations in the two logical partial orders give the canonical maps for the initial and terminal objects in *Sets*.

Dualities	Subset logic	Partition logic
'Elements'	Elements u of S	Dits (u, u') of π
Partial order	Inclusion $S \subseteq T$	$\sigma \precsim \pi$: $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$
Canonical map	$S \to T$	$\pi \to \sigma$
Top of partial order	U all elements	$\mathbf{1}_U$, $\operatorname{dit}(\mathbf{1}_U) = U^2 - \Delta$, all dits
Bottom of partial order	$\emptyset$ no elements	$\mathbf{0}_U$, $\operatorname{dit}(\mathbf{0}_U) = \emptyset$, no dits
Extremal objects *Sets*	$\emptyset \subseteq U$, $\emptyset \to U$	$\mathbf{0}_U \precsim \mathbf{1}_U$, $U \to 1$

Table 1: Elements and distinctions in the dual logics

Another simple application of the elements & distinctions analysis is the construction of the canonical surjection and canonical injection in the epi-mono factorization of any set function: $f : X \to Y$. The data in the function provide the *coimage* (or inverse-image) partition $f^{-1} = \left\{f^{-1}(y) : y \in f(X)\right\}$ on X and the *image* subset $f(X) \subseteq Y$. Since f^{-1} is refined by the discrete partition on X, $f^{-1} \precsim \mathbf{1}_X$, the induced surjection is the canonical map $X \twoheadrightarrow f(X)$. The inclusion $f(X) \subseteq Y$ induces the injection $f(X) \rightarrowtail Y$ and the epi-mono factorization of f is: $f : X \to Y = X \twoheadrightarrow f(X) \rightarrowtail Y$.

The thesis of the logical theory of canonical maps in *Sets* is that the canonical surjections are defined by refinement-induced maps from the logic of partitions lattice, the canonical injections are defined by inclusion-induced maps from the Boolean logic of subsets lattice, and the other canonical maps, e.g., the canonical factor maps of the universal constructions in *Sets*, are defined by compositions of these canonical maps.

1.4 Quantitative measures of elements & distinctions

The quantitative (normalized) counting measure of the elements in a subset gives the classical Laplace-Boole notion of 'logical' probability.

The quantitative (normalized) counting measure of the distinctions in a partition gives the notion of logical entropy that underlies the Shannon notion of entropy (which is not a measure in the sense of measure theory) ([2]; [5]; [6]).

That realizes the idea expressed in Gian-Carlo Rota's Fubini Lectures [10] (and in his lectures at MIT), where he noted that in view of duality between partitions and subsets, the "lattice of partitions plays for information the role that the Boolean algebra of subsets plays for size or probability" [7, p. 30] or symbolically:

$$\frac{\text{information}}{\text{partitions}} \approx \frac{\text{probability}}{\text{subsets.}}$$

Since "Probability is a measure on the Boolean algebra of events" that gives quantitatively the "intuitive idea of the size of a set", we may ask by "analogy" for some measure to capture a property for a partition like "what size is to a set." [10, p. 67] The answer is the number of distinctions or dits. The logical entropy $h(\pi) = \frac{|\operatorname{dit}(\pi)|}{|U \times U|}$ of a partition π on a non-empty finite set U is that measure on the lattice of partitions on U , i.e., the normalized counting measure on the isomorphic lattice of partition relations (= ditsets), the binary relations that are the complements of the equivalence relations on $U \times U$. Since the logical entropy $h(\pi)$ is also a normalized measure, it has a probability interpretation, i.e., $h(\pi)$ is the probability that in two independent draws from U, one will get a distinction of π, i.e., π distinguishes, just as $\Pr(S)$ is interpreted as the probability that in one draw from U, one will get an element of $S \subseteq U$, i.e., S occurs.[3]

Duality in quant. measures	Subset logic	Partition logic
'Outcomes'	Elements u of U	Distinctions $(u, u') \in U \times U$
'Events'	Subsets S of U	Partitions π of U
'Event occurs'	$u \in S$	$(u, u') \in \operatorname{dit}(\pi)$
Logical measure	$\Pr(S) = \frac{\|S\|}{\|U\|}$	$h(\pi) = \frac{\|\operatorname{dit}(\pi)\|}{\|U \times U\|}$
Interpretation	Prob. event S occurs	Prob. partition π distinguishes

Table 2: Logical measures on elements and distinctions

2 The Elements & Distinctions Treatment of Morphisms in *Sets*

2.1 Set functions transmit elements and reflect distinctions

The duality between elements ("its") of a subset and distinctions ("dits") of a partition already appears in the very notion of a function between sets. The concepts of elements and distinctions provide the natural notions to specify the binary relations, i.e., subsets $R \subseteq X \times Y$, that define functions $f : X \to Y$.

A binary relation $R \subseteq X \times Y$ *transmits elements* if for each element $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.

A binary relation $R \subseteq X \times Y$ *reflects elements* if for each element $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.

A binary relation $R \subseteq X \times Y$ *transmits distinctions* if for any pairs (x, y) and (x', y') in R, if $x \neq x'$, then $y \neq y'$.

A binary relation $R \subseteq X \times Y$ *reflects distinctions* if for any pairs (x, y) and (x', y') in R, if $y \neq y'$, then $x \neq x'$.

The dual role of elements and distinctions can be seen if we translate the usual characterization of the binary relations that define functions into the elements-and-distinctions language. In the usual treatment, a binary relation $R \subseteq X \times Y$ defines a *function* $X \to Y$ if it is defined everywhere on X and is single-valued. But "being defined everywhere" is the same as transmitting (or "preserving") elements, and being single-valued is the same as reflecting distinctions so the more natural definition is:

a binary relation R is a *function* if it transmits elements and reflects distinctions.

What about the other two special types of relations, i.e., those which transmit (or preserve) distinctions or reflect elements? The two important special types of functions are the injections and surjections, and they are defined by the other two notions:

a function is *injective* if it transmits distinctions, and
a function is *surjective* if it reflects elements.

[3] As a measure, the notions of simple, compound, conditional, and mutual logical entropy satsify the usual Venn diagram relationships. Shannon [11] defined those compound notions for his entropy to also satisfy the Venn diagram relations even though Shannon entropy is not a measure. This is accounted for by the fact that there is a uniform but non-linear dit-to-bit transform from the logical entropy formulas to the Shannon entropy formulas that preserves Venn diagrams. [5] Thus the notion of logical entropy gives the basic logical theory of information and the Shannon's "mathematical theory of commuication" [11] is a transform specifically for the theory of coding and communication.

Given a set function $f : X \to Y$ with *domain* X and *codomain* Y, a subset of the codomain Y is determined as the *image* $f(X) \subseteq Y$, and a partition on the domain X is determined as the *coimage* or *inverse-image* $\{f^{-1}(y)\}_{y \in f(X)}$. It might also be noted that the empty set of ordered pairs $\emptyset \times Y$ satisfies the definition of a function $\emptyset \to Y$; its image is the empty subset $\emptyset \subseteq Y$ and its coimage or inverse-image is the empty partition on $\emptyset$.

2.2 Abstracting to arrow-theoretic definitions

One of our themes is that the concepts of elements and distinctions unpack and analyze the category theoretic concepts in the basic 'ur-category' *Sets*, and they are abstracted into purely arrow-theoretic definitions in abstract category theory. For instance, the elements & distinctions definitions of injections and surjections yield "arrow-theoretic" characterizations which can then be applied in any category to provide the usual category-theoretic dual definitions of monomorphisms (injections for set functions) and epimorphisms (surjections for set functions).

Two set functions $f, g : X \rightrightarrows Y$ are different, i.e., $f \neq g$, if there is an element x of X such that their values $f(x)$ and $g(x)$ are a distinction of Y, i.e., $f(x) \neq g(x)$. Hence if f and g are followed by a function $h : Y \to Z$, then the compositions $hf, hg : X \to Y \to Z$ must be different if h *preserves distinctions* (so that the distinction $f(x) \neq g(x)$ is preserved as $hf(x) \neq hg(x)$), i.e., if h is injective. Thus in the category of sets, h being injective is characterized by: for any $f, g : X \rightrightarrows Y$, "$f \neq g$ implies $hf \neq hg$" or equivalently, "$hf = hg$ implies $f = g$" which is the general category-theoretic definition of a *monomorphism* or *mono*.

In a similar manner, if we have functions $f, g : X \rightrightarrows Y$ where $f \neq g$, i.e., where there is an element x of X such that their values $f(x)$ and $g(x)$ are a distinction of Y, then suppose the functions are preceded by a function $h : W \to X$. Then the compositions $fh, gh : W \to X \to Y$ must be different if h *reflects elements* (so that the element x where f and g differ is sure to be in the image of h), i.e., if h is surjective. Thus in the category of sets, h being surjective is characterized by: for any $f, g : X \rightrightarrows Y$, "$f \neq g$ implies $fh \neq fg$" or "$fh = gh$ implies $f = g$" which is the general category-theoretic definition of an *epimorphism* or *epi*.

Hence the dual interplay of the notions of elements & distinctions can be seen as yielding the arrow-theoretic characterizations of injections and surjections which are lifted into the general categorical dual definitions of monomorphisms and epimorphisms.

2.3 Duality interchanges elements & distinctions

The reverse-the-arrows duality of category theory is the abstraction from the reversing of the roles of elements & distinctions (or its & dits) in dualizing *Sets* to $Sets^{op}$. That is, a concrete morphism in $Sets^{op}$ is a binary relation, which might be called a *cofunction*, that preserves distinctions and reflects elements–instead of preserving elements and reflecting distinctions. Equivalently, when we reverse the direction of a binary relation defining a function, we just interchanged "reflects" and "preserves" (or "transmits"). Thus with every binary relation $f \subseteq X \times Y$ that is a function $f : X \to Y$, there is a binary relation $f^{op} \subseteq Y \times X$ that is a cofunction $f^{op} : Y \to X$ in the opposite direction.

The reverse-the-arrows duality can also be applied within *Sets*. For the universal constructions in *Sets*, the interchange in the roles of elements and distinctions interchanges each construction and its dual: products and coproducts, equalizers and coequalizers, and in general limits and colimits. That is then abstracted to make the reverse-the-arrows duality in abstract category theory.

This begins to illustrate our theme that the language of elements & distinctions is the conceptual language in which the category of sets and functions is written, and abstract category theory gives the abstract-arrows version of those definitions. Hence we turn to universal constructions for further analysis.

3 The Elements & Distinctions Analysis of Products and Coproducts

3.1 The coproduct in *Sets*

Given two sets X and Y in *Sets*, the idea of the *coproduct* is to create the set with the maximum number of elements starting with X and Y. Since X and Y may overlap, we must make two copies of the elements in the intersection. Hence the relevant operation is not the union of sets $X \cup Y$ but the disjoint union $X \sqcup Y$. To take the disjoint union of a set X with itself, a copy $X^* = \{x^* : x \in X\}$ of X is made so that $X \sqcup X$ can be constructed as $X \cup X^*$. In a similar manner, if X and Y overlap, then $X \sqcup Y = X \cup Y^*$. Then the inclusions $X, Y \subseteq X \sqcup Y$, give the canonical injections $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$.

The universal mapping property for the coproduct in *Sets* is that given any other 'cocone' of maps $f : X \to Z$ and $g : Y \to Z$, there is a unique map $f \sqcup g : X \sqcup Y \to Z$ such that $X \stackrel{i_X}{\to} X \sqcup Y \stackrel{f \sqcup g}{\to} Z = X \stackrel{f}{\to} Z$ and $Y \stackrel{i_Y}{\to} X \sqcup Y \stackrel{f \sqcup g}{\to} Z = Y \stackrel{g}{\to} Z$.

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & X \sqcup Y \arrow[d, "\exists! \, f \sqcup g"] & Y \arrow[l, "i_Y"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Coproduct diagram

From the data $f : X \to Z$ and $g : Y \to Z$, we need to construct the unique factor map $X \sqcup Y \to Z$. The map f contributes the coimage partition f^{-1}on X and g contributes the coimage partition g^{-1} on Y. The disjoint union of these two partitions on different sets gives the partition $f^{-1} \sqcup g^{-1}$ on the disjoint union $X \sqcup Y$ of the sets. That partition is refined by the discrete partition on the disjoint union, i.e., $f^{-1} \sqcup g^{-1} \precsim \mathbf{1}_{X \sqcup Y}$. Hence each element $w \in X \sqcup Y$ (i.e., each block of $\mathbf{1}_{X \sqcup Y}$) is contained in a unique block of the form $f^{-1}(z)$ for some $z \in f(X)$ or a block of the form $g^{-1}(z)$ for some $z \in g(Y)$, so the map $f \sqcup g$ first takes w to the z depending on the case. That refinement-defined map is the surjection of $X \sqcup Y$ onto $f(X) \cup g(Y) \subseteq Z$ and that inclusion defines the injection that completes the definition of the factor map $f \sqcup g : X \sqcup Y \to Z$.

3.2 The product in *Sets*

Given two sets X and Y in *Sets*, the idea of the *product* is to create the set with the maximum number of distinctions starting with X and Y. The product in *Sets* is usually constructed as the set of ordered pairs in the Cartesian product $X \times Y$. But to emphasize the point about distinctions, we might employ the same trick of 'marking' the elements of Y, particularly when $Y = X$, with an asterisk. Then an alternative construction of the product in *Sets* is the set of *unordered* pairs $X \boxtimes Y = \{\{x, y^*\} : x \in X; y^* \in Y^*\}$ which in the case of $Y = X$ would be $X \boxtimes X = \{\{x, x^*\} : x \in X; x^* \in X^*\}$. This alternative construction of the product (isomorphic to the Cartesian product) emphasizes the distinctions formed from X and Y so the ordering in the ordered pairs of the usual construction $X \times Y$ is only a way to make the same distinctions.

The set X defines a partition π_X on $X \times Y$ whose blocks are $B_x = \{(x, y) : y \in Y\} = \{x\} \times Y$ for each $x \in X$, and Y defines a partition π_Y whose blocks are $B_y = \{(x, y) : x \in X\} = X \times \{y\}$ for each $y \in Y$. Since $\pi_X, \pi_Y \precsim \mathbf{1}_{X \times Y}$, the induced maps (surjections if X and Y are non-empty) are the canonical projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$.

The universal mapping property for the product in *Sets* is that given any other 'cone' of maps $f : Z \to X$ and $g : Z \to Y$, there is a unique map $[f, g] : Z \to X \times Y$ such that $Z \stackrel{[f,g]}{\to} X \times Y \stackrel{p_X}{\to} X = Z \stackrel{f}{\to} X$ and $Z \stackrel{[f,g]}{\to} X \times Y \stackrel{p_Y}{\to} Y = Z \stackrel{g}{\to} Y$.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & \exists! \downarrow^{[f,g]} & g \searrow & \\ X & \overset{p_X}{\longleftarrow} & X \times Y & \overset{p_Y}{\longrightarrow} & Y \end{array}$$

Product diagram

From the data $f : Z \to X$ and $g : Z \to Y$, we need to construct the unique factor map $[f, g] : Z \to X \times Y$. The map f contributes the coimage f^{-1} partition on Z and g contributes the coimage g^{-1} partition on Z so we have the partition join $f^{-1} \vee g^{-1}$ (in the lattice of partitions on Z) whose blocks have the form $f^{-1}(x) \cap g^{-1}(y)$. The discrete partition $\mathbf{1}_Z$ refines $f^{-1} \vee g^{-1}$ as partitions on Z so for each singleton $\{z\}$, there is a block of the form $f^{-1}(x) \cap g^{-1}(y)$ containing z and this defines the canonical map $Z \to f(Z) \times g(Z) \subseteq X \times Y$ where $z \longmapsto (x, y)$, and that inclusion gives the injection $f(Z) \times g(Z) \to X \times Y$ that completes the construction of the factor map $[f, g] : Z \to X \times Y$.

3.3 Example: How duality interchanges products and coproducts

To show how duality interchanges universal constructions and their duals, consider the product in the category $Sets^{op}$ of sets and cofunctions. In *Sets*, the product $X \times Y$ has the maximum number of distinctions from X and Y, so in $Sets^{op}$, the product set $X \times^{op} Y$ has the maximum number of elements (interchanging distinctions and elements), i.e., $X \times^{op} Y = X \sqcup Y$. And each cofunction (binary relation that preserves distinctions and reflects elements in one direction) is in fact a binary relation in the other direction (interchange "reflects" and "preserves") that reflects distinctions and preserves elements, i.e., a function. Hence the two cofunctions $p_X^{op} : X \times^{op} Y \to X$ and $p_Y^{op} : X \times^{op} Y \to Y$ that reflect distinctions, are the same as two functions that preserve distinctions, i.e., the two injections $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$. For the UMP, the two cofunctions $f^{op} : Z \to X$ and $g^{op} : Z \to Y$ in $Sets^{op}$ are two functions $f : X \to Z$ and $g : Y \to Z$ in the opposite direction in *Sets*, and the unique cofunction $[f^{op}, g^{op}] : Z \to X \times^{op} Y$ satisfying the commutativity properties in $Sets^{op}$ is the same as the function $f \sqcup g : X \sqcup Y \to Z$ satisfying the opposite commutativity properties in *Sets*. Thus the product in $Sets^{op}$ is the same as the coproduct in *Sets*. Hence the concrete duality of interchanging elements and distinction–or equivalently for binary relations, interchange "reflects" and "preserves" (or "transmits")–just interchanges the dual universal constructions of products and coproducts.

4 The Elements & Distinctions Analysis of Equalizers and Coequalizers

4.1 The coequalizer in *Sets*

For the equalizer and coequalizer, the data is not just two sets but two parallel maps $f, g : X \rightrightarrows Y$. Then each element $x \in X$, gives us a pair $f(x)$ and $g(x)$ so we take the equivalence relation $\sim$ defined on Y that is generated by $f(x) \sim g(x)$ for any $x \in X$. Then the coequalizer is the quotient set $C = Y/\sim$. When $\sim$ is represented as a partition on Y, then it is refined by the discrete partition $\mathbf{1}_Y$ on Y, and that refinement defines the canonical surjective map $can. : Y \to Y/\sim$.

For the universality property, let $h : Y \to Z$ be such that $hf = hg$. Then we need to show there is a unique refinement/inclusion-defined map $h^* : Y/\sim \to Z$ such that $h^* can. = h$.

$$\begin{array}{ccccc} X & \underset{g}{\overset{f}{\rightrightarrows}} & Y & \overset{can.}{\to} & Y/\sim \\ & & & \searrow^{h} & \exists! \downarrow^{h^*} \\ & & & & Z \end{array}$$

Coequalizer diagram

We already have one partition $\sim$ on Y which was generated by $f(x) \sim g(x)$. Since $hf = hg$, we know that $hf(x) = hg(x)$ so the coimage h^{-1} has to *at least* identify $f(x)$ and $g(x)$ (and perhaps identify other elements) which means that $h^{-1} \precsim \sim$ as partitions on Y. Hence for each element of $Y/\sim$, i.e., each block b in the partition $\sim$, there is a unique block $h^{-1}(z)$ containing that block, so the induced map is $h^{*}(b) = z$ onto $h(Y) \subseteq Z$ and the inclusion defines the injection $h(Y) \to Z$ which completes the construction of $h^{*} : Y/\sim \to Z$.

4.2 The equalizer in *Sets*

The data for the equalizer construction is the same two parallel maps $f, g : X \rightrightarrows Y$. The equalizer is the $E = \{x \in X : f(x) = g(x)\} \subseteq X$ so the map induced by that inclusion is the canonical map $can. : E \to X$.

The universal property is that for any other map $h : Z \to X$ such that $fh = gh$, then $\exists! h_{*} : Z \to E$ such that $h_{*} can. = h$.

$$\begin{tikzcd} E \arrow[r, "can."] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ Z \arrow[u, "\exists!" ', "h_*"] \arrow[ur, "h"] & & \end{tikzcd}$$

Equalizer diagram

For each $x \in h(Z)$, $fh(z) = gh(z)$ implies $h(z) = x \in E$ so $h(Z) \subseteq E$. The coimage partition $h^{-1} = \left\{h^{-1}(x) : x \in h(Z)\right\}$ on Z is refined by the discrete partition on Z and that as usual defines the surjection of $Z \to h(Z)$ and then $h(Z) \subseteq E$ induces the injection $h(Z) \to Z$ to complete the definition of the canonical factor map $h_{*} : Z \to E$.

5 The Elements & Distinctions Analysis of Cartesian and Co-Cartesian Squares

5.1 The pushout or co-Cartesian square in *Sets*

It is a standard theorem of category theory that if a category has products and equalizers, then it has all limits, and if it has coproducts and coequalizers, then it has all colimits. Since we have presented the elements & distinctions analysis of the canonical maps for products and coproducts, and for equalizers and coequalizers, the analysis extends to all limits and colimits. Hence we have shown that the logical characterization of canonical maps in *Sets* satisfies Marquis's criteria:

> 1. Morphisms that are part of the data of a limit are canonical morphisms; for instance, the projection morphisms that are part of the notion of a product;
> 2. The unique morphism from a cone to a limit determined by a universal property is a canonical morphism: and
> 3. In particular, the unique isomorphism that arise between two candidates for a limit is a canonical morphism. [9, p. 101]

However, the theme would be better illustrated by considering some more complicated limits and colimits such as Cartesian and co-Cartesian squares, i.e., pullbacks and pushouts.

For the pushout or co-Cartesian square, the data are two maps $f : Z \to X$ and $g : Z \to Y$ so we have the two parallel maps $Z \xrightarrow{f} X \xrightarrow{i_X} X \sqcup Y$ and $Z \xrightarrow{g} Y \xrightarrow{i_Y} X \sqcup Y$ and then we can take their coequalizer C formed by the equivalence relation $\sim$ on the common codomain $X \sqcup Y$ which is the equivalence relation generated by $x \sim y$ if there is a $z \in Z$ such that $f(z) = x$ and $g(z) = y$. The canonical maps $X \to X \sqcup Y/\sim$ and $Y \to X \sqcup Y/\sim$ are just the canonical injections into the disjoint union followed by the canonical map of the coequalizer construction analyzed above. As the composition of a canonical injection with a canonical surjection, those canonical maps need not be either injective or surjective.

$$\begin{array}{ccccc}
Z & \stackrel{f}{\rightarrow} & X & = & X \\
{}^{g}\downarrow & \searrow & \downarrow^{can..} & & \\
Y & \stackrel{can.}{\rightarrow} & C = X \sqcup Y/\sim & & \downarrow^{h} \\
\shortparallel & & & \exists! \searrow^{h^*} & \\
Y & & \stackrel{h'}{\rightarrow} & & U
\end{array}$$

Pushout or co-Cartesian square diagram

For the universal mapping property, consider any $h : X \rightarrow U$ and $h' : Y \rightarrow U$ such that $hf = h'g$. Then h^{-1} is a partition on X and h'^{-1} is a partition on Y so let $h^{-1} \sqcup h'^{-1}$ be the disjoint union partition on $X \sqcup Y$. The condition that for any $z \in Z$, $hf(z) = h'g(z) = u$ for some $u \in U$ means that $h^{-1} \sqcup h'^{-1}$ must make at least the identifications of the coequalizer (and perhaps more) so that $h^{-1} \sqcup h'^{-1}$ is refined by $\sim$ as partitions on $X \sqcup Y$. Since $h^{-1} \sqcup h'^{-1} \precsim \sim$ so each block b in $\sim$ is contained in a block of the form $h^{-1}(u)$ for some u or a block of the form $h'^{-1}(u)$ for some u. Hence that block b of $\sim$ is mapped by h^* to the appropriate u depending on the case which defines the surjection from $X \sqcup Y/\sim$ to $h(X) \cup h'(Y) \subseteq U$ and the inclusion defines the injection to complete the definition of the canonical factor map $h^* : C = X \sqcup Y/\sim \rightarrow U$.

5.2 The pullback or Cartesian square in *Sets*

For the Cartesian square or pullback, the data are two maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$. We then have two parallel maps $X \times Y \rightrightarrows Z$ (the projections followed by f or g) so we take the pullback as their equalizer E. The canonical maps $E \rightarrow X$ and $E \rightarrow Y$ are the compositions of the canonical injective map $E \rightarrow X \times Y$ followed by the canonical projections $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$. As the composition of a canonical injection with a canonical surjection, those canonical maps need not be either injective or surjective.

$$\begin{array}{ccccc}
U & & \stackrel{h}{\rightarrow} & & X \\
 & \exists! \searrow^{h_*} & & & \shortparallel \\
\downarrow^{h'} & & E \subseteq X \times Y & \stackrel{can.}{\rightarrow} & X \\
 & & \downarrow^{can.} & & \downarrow^{f} \\
Y & = & Y & \stackrel{g}{\rightarrow} & Z
\end{array}$$

Pullback or Cartesian square diagram

For the universality property, consider any other maps $h : U \rightarrow X$ and $h' : U \rightarrow Y$ such that $fh = gh'$. Hence $h'(u)$ and $h(u)$ are elements such that $f(h(u)) = g(h'(u))$ so $(h(u), h'(u)) \in E$ and thus for the images, there is the inclusion $h(U) \times h'(U) \subseteq E$. Now h contributes the coimage partition h^{-1} on U and h' contributes the coimage partition h'^{-1} on U and the join $h^{-1} \vee h'^{-1}$ is refined by the discrete partition on U. Hence each $u \in U$ is contained in a unique block $h^{-1}(x) \cap h'^{-1}(y)$ of the join so the refinement-induced canonical map $U \rightarrow h(U) \times h'(U) \subseteq E$ is defined by $u \longmapsto (x, y)$ and the inclusion-defined injection $h(U) \times h'(U) \rightarrow E$ completes the definition of the canonical factor map $h_* : U \rightarrow E$.

6 Example: A more complex canonical map

Marquis [9] gives the standard examples of canonical maps that arise from limits and colimits but also mentions a more complex example that will be analyzed. Let $\mathcal{C}$ be a category with finite products, finite coproducts, and a null object (an object that is both initial and terminal). Then a canonical morphism can be constructed from the coproduct of two (or any finite number of) objects to the products of the objects: $X \sqcup Y \to X \times Y$. In such a category abstractly specified, the map could be constructed from the 'atomic' canonical morphisms that are already given by the arrow-theoretic definitions of products, coproducts, and the null object. But the its & dits analysis shows how all these 'atomic' canonical morphisms and their 'molecular' compositions are not just assumed but are constructed in *Sets* or *Sets*-based categories according to the logical theory of canonical maps.

There is a simple *Sets*-based category that has finite products, finite products, and a null object, namely the category $Sets_*$ of pointed sets where the objects are sets with a designated element (or basepoint), e.g., (X, x_0) with $x_0 \in X$, and the morphisms are set functions that preserve the basepoints. The designation of the basepoint can be given by a set map $1 \overset{x_0}{\to} X$ in *Sets* which is taken as part of the structure and is thus assumed canonical in $Sets_*$. A basepoint preserving map $(X, x_0) \to (Y, y_0)$ is a set map $X \to Y$ in *Sets* so that the following diagram commutes:

$$\begin{array}{ccc} 1 & & \\ \downarrow^{x_0} & \searrow^{y_0} & \\ X & \to & Y \end{array} .$$

Hence $Sets_*$ can also be seen as the slice category $1/Sets$ of *Sets* under 1.

The null object is 'the' one-point set 1 and instead of assuming the canonical morphisms that make it both initial and terminal, we need to construct them using the its & dits analysis. We have already seen that the refinement relation $\mathbf{0}_X \precsim \mathbf{1}_X$ induces the unique map $X \to 1$ that makes 1 the terminal object in *Sets*. And since $1 \overset{x_0}{\to} X \to 1 = 1 \overset{id.}{\to} 1$, it is also the terminal object in $Sets_*$. Moreover, the basepoint in (Y, y_0) is given by the structurally canonical map $1 \overset{y_0}{\to} Y$ and since $1 \overset{id.}{\to} 1 \overset{y_0}{\to} Y = 1 \overset{y_0}{\to} Y$, that map $1 \overset{y_0}{\to} Y$ is the unique map that makes 1 also the initial object in $Sets_*$. Hence in $Sets_*$, there is always a canonical map formed by the composition: $X \to 1 \to Y = X \to Y$ (called the *zero arrow*).

To build up the its & dits analysis of the canonical morphism $X \sqcup_* Y \to X \times_* Y$ from the coproduct to the product in $Sets_*$, we begin with the construction of the coproduct $X \sqcup_* Y$ which is just the pushout in *Sets* of the two canonical basepoint maps:

$$\begin{array}{ccccc} 1 & \overset{x_0}{\to} & X & = & X \\ y_0 \downarrow & \searrow & \downarrow^{can..} & & \\ Y & \overset{can.}{\to} & X \sqcup_* Y = X \sqcup Y/\sim & & \downarrow^{h} \\ \Vert & & & \exists! \searrow^{h_*} & \\ Y & & \overset{h'}{\to} & & U \end{array}$$

Since the only points in X and Y that are the image of an elements in 1 are the basepoints, the equivalence relation $\sim$ only identifies the basepoints x_0 and y_0. Hence $X \sqcup_* Y$ is like $X \sqcup Y$ except that the two basepoints are identified in the quotient $X \sqcup_* Y = X \sqcup Y/\sim$ and that block identifying the basepoints is the basepoint of $X \sqcup_* Y$. Then for any two set maps $h : X \to U$ and $h' : Y \to U$ that are also $Sets_*$ morphisms (i.e., preserve basepoints), there is a unique canonical factor map $h_* : X \sqcup_* Y \to U$ by the UMP for the pushout in *Sets* to make the triangles commute (and thus preserve basepoints). Hence $X \sqcup_* Y$ is the coproduct in $Sets_*$.

In a similar manner, one shows that the product $X \times_* Y$ in $Sets_*$ is just the product $X \times Y$ in *Sets* with $\langle x_0, y_0 \rangle$ as the basepoint. Using the UMP of the product $X \times_* Y$ in $Sets_*$, the two $Sets_*$ maps $1_X : X \to X$ and the canonical $X \to 1 \to Y$, we have the unique canonical factor map $X \to X \times_* Y$ in $Sets_*$ and similarly for $Y \to X \times_* Y$ in $Sets_*$.

Then we put all the canonical maps together and use the UMP for the coproduct in $Sets_*$ to construct the desired canonical map: $X \sqcup_* Y \to X \times_* Y$ in $Sets_*$.

$$\begin{tikzcd} X \arrow[r, "can."] \arrow[dr, "can."] & X \sqcup_* Y \arrow[d, "\exists!\ can."] & Y \arrow[l, "can."'] \arrow[dl, "can."'] \\ & X \times_* Y & \end{tikzcd}$$

Coproduct diagram in $Sets_*$

This example shows how in a $Sets$-based category like $Sets_*$, the given canonical maps for the structured sets (i.e., the basepoint maps $1 \xrightarrow{x_0} X$) are combined with the canonical maps defined by the its & dits analysis in $Sets$ to give the canonical morphisms in the $Sets$-based category. In abstract category theory, as in the case of a category $\mathcal{C}$ which is assumed to have finite products, finite coproducts, and a null object, the 'atomic' canonical morphisms are all given as part of the assumed UMPs for products, coproducts, and the null object which are then composed to define other 'molecular' canonical morphisms.

7 Concluding Remarks

The "logical" in the logical theory of canonicity refers to the two dual mathematical logics: the Boolean logic of subsets and the logic of partitions. Note that from the mathematical viewpoint, the Boolean logic of subsets and the logic of partitions have *equal* intertwining roles in the whole analysis. Normally, we might say that "subsets" and "partitions" are category-theoretic duals, but we have tried to show a more fundamental analysis based on "elements & distinctions" or "its & dits" that are the building blocks of subsets and partitions and that underlie the duality in $Sets$.

The dual interplay of elements and distinctions explains morphisms, duality, canonicity, and universal constructions in $Sets$, which generalizes to other $Sets$-based concrete categories and which is abstracted in abstract category theory. Our focus here is the E&D treatment of canonicity.

- Each construction starts with certain data.
- When that data is sufficient to define inclusions in an associated subset lattice or refinements in an associated partition lattice, then the resulting injections and surjections (and their compositions) are canonical.
- That is the logical theory of canonicity

This suggests that the dual notions of elements & distinctions (its & dits) have some broader significance. One possibility is they are respectively mathematical versions of the old metaphysical concepts of matter (or substance) and form (as in in-form-ation). The matter versus form idea [1] can be illustrated by comparing the two lattices of subsets and partitions on a set–the two lattices that we saw defined the canonical morphisms and canonical factor maps in $Sets$-based categories.

For $U = \{a, b, c\}$, start at the bottom and move towards the top of each lattice.

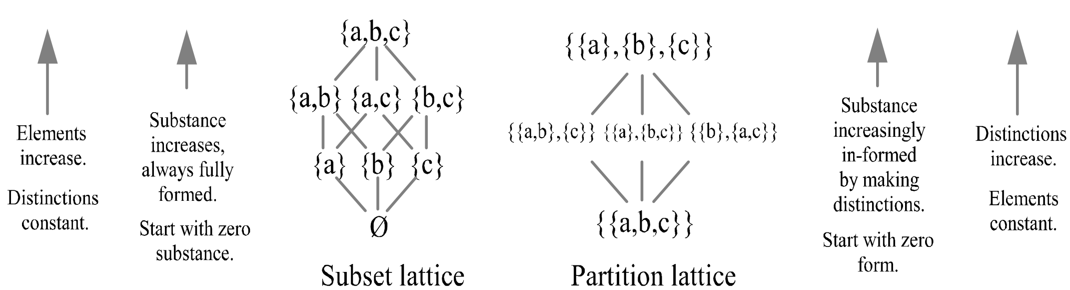

Figure 1: Moving up the subset and partition lattices.

At the bottom of the Boolean subset lattice is the empty set $\emptyset$ which represents no substance (no 'its'). As one moves up the lattice, new elements of substance, new "its", are created that are always fully formed until finally one reaches the top, the universe U. Thus new substance is created in moving up the lattice but each element is fully formed and thus distinguished from the other elements.

At the bottom of the partition lattice is the indiscrete partition or "blob" $\mathbf{0}_U = \{U\}$ (where the universe set U makes one block) which represents all the substance or matter but with no distinctions to in-form the substance (no 'dits'). As one moves up the lattice, no new substance is created but distinctions are created that in-form the indistinct elements as they become more and more distinct. Finally one reaches the top, the discrete partition $\mathbf{1}_U$, where all the elements of U have been fully formed. A partition combines indefiniteness (within blocks) and definiteness (between blocks). At the top of the partition lattice, the discrete partition $\mathbf{1}_U = \{\{u\} : \{u\} \subseteq U\}$ is the result making all the distinctions to eliminate any indefiniteness. Thus one ends up at essentially the "same place" (universe U of fully formed entities) either way, but by two totally different but dual 'creation stories':

- creating elements (as in creating fully-formed matter out of nothing) versus
- creating distinctions (as in starting with a totally undifferentiated matter and then, in a 'big bang,' start making distinctions, e.g., breaking symmetries, to give form to the matter).

Moreover, we have seen that:

- the quantitative increase in substance (normalized number of elements) moving up in the subset lattice is measured by logical or Laplacian probability, and
- the quantitative increase in form (normalized number of distinctions) moving up in the partition lattice is measured by logical information or logical entropy ([2]; [5]).

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