The Logical Theory of Canonicity: The Elements & Distinctions Analysis of Morphisms, Duality, Canonicity, and Universal Constructions in *Sets*

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Introduction: Elements & Distinctions

- Purpose is to describe the "elements & distinctions" analysis of morphisms, duality, canonicity, and universality in *Sets*, with a focus on canonicity.
- Start with the duality of subsets and partitions (= quotient sets = equivalence relations).
- What is the dual notion to "elements of a subset"?
- It is distinctions (pair of elements in different blocks) of a partition.
- Thus "elements & distinctions" (or "its and dits") are basic conceptual building blocks in subsets & partitions.

E&D analysis of morphisms: I

- A binary relation $R \subseteq X \times Y$ *transmits (or preserves) elements* if for each element $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.
- A binary relation $R \subseteq X \times Y$ *reflects elements* if for each element $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.
- A binary relation $R \subseteq X \times Y$ *transmits (or preserves) distinctions* if for any pairs (x, y) and (x', y') in R, if $x \neq x'$, then $y \neq y'$.
- A binary relation $R \subseteq X \times Y$ *reflects distinctions* if for any pairs (x, y) and (x', y') in R, if $y \neq y'$, then $x \neq x'$.
- A set function $f : X \to Y$ is usually characterized as being defined everywhere and single-valued, but:

E&D analysis of morphisms: II

- "Defined everywhere" is the same as "transmits elements" and
- "Being single-valued" is the same as "reflecting distinctions."

Binary relation is a *function*
iff it transmits elements and reflects distinctions.

- What about the other two notions of "reflecting elements" and "transmitting distinctions":
  - A function $f : X \to Y$ is injective iff it transmits distinctions; and
  - A function $f : X \to Y$ if surjective iff it reflects elements.
- The Es&Ds provide the natural language to define the morphisms in *Sets* and the special types of injections and surjections.

Duality interchanges Es & Ds: I

- The reverse-the-arrows duality of category theory is the abstraction from the reversing of the roles of elements & distinctions in dualizing *Sets* to $Sets^{op}$.
- That is, a *concrete* morphism in $Sets^{op}$ is a binary relation, which might be called a *cofunction*, that preserves distinctions and reflects elements–instead of preserving elements and reflecting distinctions.
- Thus with every binary relation $f \subseteq X \times Y$ that is a function $f : X \to Y$, there is a binary relation $f^{op} \subseteq Y \times X$ that is a cofunction $f^{op} : Y \to X$.

Duality interchanges Es & Ds: II

- For the universal constructions in *Sets*, the interchange in the roles of elements and distinctions interchanges each construction and its dual: products and coproducts, equalizers and coequalizers, and in general limits and colimits.
- That is then abstracted to make the reverse-the-arrows duality in abstract category theory.
- This begins to illustrate our theme that the language of elements & distinctions is the conceptual language in which the category of sets and functions is written, and abstract category theory gives the abstract-arrows version of those definitions. Hence we turn to the E&D treatment canonicity.

E & D analysis of *canonical* maps: I

- Category theory has a mathematical notion of naturality but only an intuitive notion of canonical maps.
- In Marquis, Jean-Pierre. 2017. "Canonical Maps." In *Categories for the Working Philosopher*, edited by Elaine Landry, 90–112. Oxford UK: Oxford University Press, the mathematical philosopher, J-P Marquis, raised the question of defining canonicity.
- Marquis gives the intuitive idea (maps defined "without any arbitrary decision") and criteria stated in terms of limits (and thus dually for colimits).

E & D analysis of *canonical* maps: II

We are now in a position to circumscribe more precisely what we want to include in the notion of canonical morphisms or maps.

1. Morphisms that are part of the data of a limit are canonical morphisms; for instance, the projection morphisms that are part of the notion of a product;

2. The unique morphism from a cone to a limit determined by a universal property is a canonical morphism: and

3. In particular, the unique isomorphism that arise between two candidates for a limit is a canonical morphism.

E & D analysis of *canonical* maps: III

- The *logical theory of canonicity*, based on E&D analysis, characterizes canonical maps as arising from the partial orders in the two dual logics of subsets and of partitions:
  - Boolean lattice $\wp(U)$ of subsets of U: partial order is inclusion $S \subseteq T$ for $S, T \in \wp(U)$ induces the canonical injection: $S \rightarrow T$;
  - Partition lattice $\Pi(U)$ of partitions on a non-empty U: partial order is refinement $\sigma \precsim \pi$ [i.e., dits of σ are dits of π, $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$] for $\sigma, \pi \in \Pi(U)$, which means for every block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$ and which induces the canonical surjection: $\pi \rightarrow \sigma$. Any partitional reasoning in $\Pi(U)$ assumes a non-empty U since there are no partitions on the empty set.

E & D analysis of *canonical* maps: IV

- The claim is that all “canonical” maps in *Sets* arise in either of these ways or by compositions of these maps. A claim about an intuitive notion like "canonical" cannot be proven–but it can be demonstrated.

The terminal object, initial object, and epi-mono factorization in Sets: I

- Since every partition $\pi = \{B, B', ...\}$ on U is refined by the top of the partition lattice $\Pi(U)$, the *discrete partition* $\mathbf{1}_U = \{\{u\} : u \in U\}$, i.e., $\pi \precsim \mathbf{1}_U$, there is the canonical surjection $\mathbf{1}_U \cong U \rightarrow \pi$ that takes the singleton $\{u\}$ to the unique block B such that $u \in B$.
- The bottom of the partition lattice, the indiscrete partition $\mathbf{0}_U = \{\{U\}\}$, is refined by all partitions on U, e.g., $\mathbf{0}_U \precsim \mathbf{1}_U$ so there is a canonical surjection $\mathbf{1}_U \cong U \rightarrow \mathbf{0}_U \cong 1$ ('the' one-element set). Taking U as any set in *Sets*, this canonical surjection establishes 1 as the *terminal object* in *Sets*.
- The top of the Boolean algebra $\wp(U)$ is U, so each subset $S \subseteq U$ induces the canonical injection $S \rightarrow U$.

The terminal object, initial object, and epi-mono factorization in Sets: II

- The bottom of the subset lattice, the empty set $\emptyset$, is contained in every subset of U, e.g., $\emptyset \subseteq U$ so there is a canonical injection $\emptyset \to U$. Taking U as any set in *Sets*, this canonical injection establishes $\emptyset$ as the *initial object* in *Sets*.

Dualities	Subset logic	Partition logic
'Elements'	Elements u of S	Dits (u, u') of π
Partial order	Inclusion $S \subseteq T$	$\sigma \precsim \pi$,dit $(\sigma) \subseteq$ dit (π)
Canonical map	$S \to T$	$\pi \to \sigma$
Extremal objects *Sets*	$\emptyset \subseteq U$ so $\emptyset \to U$	$\mathbf{0}_U \precsim \mathbf{1}_U$ so $U \to 1$

The terminal object, initial object, and epi-mono factorization in Sets: III

- The data in any set function: $f : X \to Y$ provide the *coimage* (or inverse-image) partition $f^{-1} = \{f^{-1}(y) : y \in f(X)\}$ on X and the *image* subset $f(X) \subseteq Y$.
  - Since f^{-1} is refined by the discrete partition on X, $f^{-1} \precsim \mathbf{1}_X$, the induced map is the canonical surjection: $X \twoheadrightarrow f(X)$.
  - The inclusion $f(X) \subseteq Y$ induces the canonical injection $f(X) \rightarrowtail Y$.
  - The epi-mono factorization of f is the composition of the canonical maps: $f : X \to Y = X \twoheadrightarrow f(X) \rightarrowtail Y$.

E&D analysis of coproducts: I

- Given two sets X and Y in *Sets*, the idea of the *coproduct* is to create the set with the maximum number of *elements* starting with X and Y.
- Since X and Y may overlap or even $Y = X$, we make a copy of Y as Y^* and define $X \sqcup Y = X \cup Y^*$.
- The inclusions $X, Y \subseteq X \sqcup Y$, give the canonical injections $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$.
- The universal mapping property (UMP) for the coproduct in *Sets* is that given any cocone of maps $f : X \to Z$ and $g : Y \to Z$, there is a unique factor map $f \sqcup g : X \sqcup Y \to Z$ such that $X \xrightarrow{i_X} X \sqcup Y \xrightarrow{f \sqcup g} Z = X \xrightarrow{f} Z$ and $Y \xrightarrow{i_Y} X \sqcup Y \xrightarrow{f \sqcup g} Z = Y \xrightarrow{g} Z$.

E&D analysis of coproducts: II

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & X \sqcup Y \arrow[d, "\exists! \; f \sqcup g"] & Y \arrow[l, "i_Y"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Coproduct diagram

- From the data $f : X \to Z$ and $g : Y \to Z$, we need to canonically construct the unique factor map $X \sqcup Y \to Z$.
- The map f contributes the coimage partition f^{-1}on X and g contributes the coimage partition g^{-1} on Y.
- They define the partition $f^{-1} \sqcup g^{-1}$ on the disjoint union $X \sqcup Y$ of the sets. That partition is refined by the discrete partition on the disjoint union, i.e., $f^{-1} \sqcup g^{-1} \precsim \mathbf{1}_{X \sqcup Y}$.

E&D analysis of coproducts: III

- Hence each element $w \in X \sqcup Y$ (i.e., each block of $\mathbf{1}_{X \sqcup Y}$) is contained in a unique block of the form $f^{-1}(z)$ for some $z \in f(X)$ or a block of the form $g^{-1}(z)$ for some $z \in g(Y)$, so the map $f \sqcup g$ first takes w to the appropriate z depending on the case.
- That refinement-defined map is the canonical surjection of $X \sqcup Y$ onto the subset lattice join $f(X) \cup g(Y) \subseteq Z$ and that inclusion defines the canonical injection that completes the canonical definition of the factor map $f \sqcup g : X \sqcup Y \to Z$.

- Given two non-empty sets X and Y in *Sets*, the idea of the *product* is to create the set with the maximum number of *distinctions* starting with X and Y.
- The product is usually constructed as the Cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$ but to emphasize the point about distinctions, we could use the isomorphic set of *unordered* pairs $X \boxtimes Y = \{\{x, y^*\} : x \in X; y^* \in Y^*\}$ which in the case of $Y = X$ would be $X \boxtimes X = \{\{x, x^*\} : x \in X; x^* \in X^*\}$.
- The set X defines a partition π_X on $X \times Y$ whose blocks are $B_x = \{(x, y) : y \in Y\} = \{x\} \times Y$ for each $x \in X$, and Y defines a partition π_Y whose blocks are $B_y = \{(x, y) : x \in X\} = X \times \{y\}$ for each $y \in Y$.

- Since $\pi_X, \pi_Y \precsim \mathbf{1}_{X\times Y}$, the induced surjections are the canonical projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$.
- The UMP for the product in *Sets* is that given any cone of maps $f : Z \to X$ and $g : Z \to Y$, there is a unique map $[f,g] : Z \to X \times Y$ such that $Z \xrightarrow{[f,g]} X \times Y \xrightarrow{p_X} X = Z \xrightarrow{f} X$ and $Z \xrightarrow{[f,g]} X \times Y \xrightarrow{p_Y} Y = Z \xrightarrow{g} Y$.

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, "{\exists! \, [f,g]}"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "p_X"'] \arrow[r, "p_Y"] & Y \end{tikzcd}$$

Product diagram

E & D analysis of products: III

- From the data $f : Z \to X$ and $g : Z \to Y$, we need to canonically construct the unique factor map $[f, g] : Z \to X \times Y$.
- The map f contributes the coimage f^{-1} partition on Z and g contributes the coimage g^{-1} partition on Z so we have the partition lattice join $f^{-1} \vee g^{-1}$ whose blocks have the form $f^{-1}(x) \cap g^{-1}(y)$.
- To define the unique factor map $[f, g] : Z \to X \times Y$, the discrete partition $\mathbf{1}_Z$ refines $f^{-1} \vee g^{-1}$ so for each singleton $\{z\}$, there is a block of the form $f^{-1}(x) \cap g^{-1}(y)$ and thus the factor map $[f, g]$ takes $z \longmapsto (x, y) \in f(Z) \times g(Z) \subseteq X \times Y$, and the induced injection completes the canonical definition of the factor map $[f, g] : Z \to X \times Y$.

E&D analysis of coequalizers: I

- For the equalizer and coequalizer, the data is not just two sets but two parallel maps $f, g : X \rightrightarrows Y$.
- Then each element $x \in X$, gives us a pair $f(x)$ and $g(x)$ so we take the equivalence relation $\sim$ defined on Y that is generated by $f(x) \sim g(x)$ for any $x \in X$.
- Then the coequalizer is the quotient set $C = Y/\sim$.
- When $\sim$ is represented as a partition on Y, then it is refined by the discrete partition $\mathbf{1}_Y$ on Y, and that refinement defines the canonical surjection $can. : Y \to Y/\sim$.
- For the UMP, let $h : Y \to Z$ be such that $hf = hg$. Then we need to show there is a unique refinement-defined map $h^* : Y/\sim \to Z$ such that $h^* can. = h$.

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{can.}{\longrightarrow} Y/\sim$$
$$\searrow^{h} \quad \exists! \downarrow h^{*}$$
$$Z$$

Coequalizer diagram

- We already have one partition $\sim$ on Y which was generated by $f(x) \sim g(x)$.
- Since $hf = hg$, we know that $hf(x) = hg(x)$ so the coimage or inverse-image h^{-1} has to *at least* identify $f(x)$ and $g(x)$ (and perhaps identify other elements) which means that $h^{-1} \precsim \sim$ in the partition lattice on Y.

- Hence for each element of $Y/\sim$, i.e., each block b in the partition $\sim$, there is a unique block $h^{-1}(z)$ containing that block, so induced canonical surjection $b \longmapsto z$ takes $Y/\sim$ to $h(Y) \subseteq Z$ and the canonical injection $h(Y) \to Z$ completes the definition of $h^{*}: Y/\sim \to Z$.

E & D analysis of equalizers: I

- For the same data $f,g : X \rightrightarrows Y$, the equalizer is the $E = \{x \in X : f(x) = g(x)\} \subseteq X$ so the map induced by that inclusion is the canonical map $can. : E \to X$.
- The UMP is that for any other map $h : Z \to X$ such that $fh = gh$,then $\exists! h_* : Z \to E$ such that $h_* can. = h$.

$$\begin{array}{ccccc} E & \overset{can.}{\longrightarrow} & X & \underset{g}{\overset{f}{\rightrightarrows}} & Y \\ \exists! \uparrow^{h_*} & h \nearrow & & & \\ Z & & & & \end{array}$$

Equalizer diagram

E & D analysis of equalizers: II

- The image of $h(Z)$ must satisfy $fh(z) = gh(z)$ for all $z \in Z$, so all the elements $h(z) \in X$ where f and g agree give $h(Z) \subseteq E$, and thus the canonical factor map h_* is induced by that inclusion.

E&D analysis of pushouts: I

- Since all limits can be constructed from products and equalizers, and all colimits can be constructed from coproducts and coequalizers, the E&D analysis of canonicity extends automatically to all limits and colimits.
- But to illustrate the more general case, we consider pushouts (and pullbacks are treated dually).
- The data for pushouts are two maps $f : Z \to X$ and $g : Z \to Y$ so we have the two parallel maps $Z \xrightarrow{f} X \xrightarrow{i_X} X \sqcup Y$ and $Z \xrightarrow{g} Y \xrightarrow{i_Y} X \sqcup Y$ and then we can take their coequalizer C formed by the equivalence relation $\sim$ on the common codomain $X \sqcup Y$ which is the equivalence relation generated by $x \sim y$ if there is a $z \in Z$ such that $f(z) = x$ and $g(z) = y$.

- The canonical maps $X \rightarrow X \sqcup Y / \sim$ and $Y \rightarrow X \sqcup Y / \sim$ are just the canonical injections into the disjoint union followed by the canonical map of the coequalizer construction analyzed above.
- For the UMP, consider any $h : X \rightarrow U$ and $h' : Y \rightarrow U$ such that $hf = h'g$.

$$
\begin{array}{ccccc}
Z & \stackrel{f}{\rightarrow} & X & = & X \\
g \downarrow & \searrow & \downarrow^{can.} & & \\
Y & \stackrel{can.}{\rightarrow} & C = X \sqcup Y / \sim & & \downarrow^{h} \\
 & & & \exists! \searrow h^{*} & \\
\parallel & & & & \\
Y & & \stackrel{h'}{\longrightarrow} & & U
\end{array}
$$

Pushout or co-Cartesian square diagram

E&D analysis of pushouts: III

- Then h^{-1} is a partition on X and h'^{-1} is a partition on Y so let $h^{-1} \sqcup h'^{-1}$ be the disjoint union partition on $X \sqcup Y$.
- The condition that for any $z \in Z$, $hf(z) = h'g(z) = u$ for some $u \in U$ means that $h^{-1} \sqcup h'^{-1}$ must make at least the identifications of the coequalizer (and perhaps more) so that $h^{-1} \sqcup h'^{-1}$ is refined by $\sim$ as partitions on $X \sqcup Y$.
- Since $h^{-1} \sqcup h'^{-1} \precsim \sim$ so each block b in $\sim$ is contained in a block of the form $h^{-1}(u)$ for some u or a block of the form $h'^{-1}(u)$ for some u.

- Hence each block b of $\sim$ is mapped by h^* to the appropriate u depending on the case which defines the canonical surjection from $X \sqcup Y / \sim$ to $h(X) \cup h'(Y) \subseteq U$ and the inclusion defines the canonical injection to complete the definition of the canonical factor map $h^* : C = X \sqcup Y / \sim \longrightarrow U$.

Extending the analysis to categories based on Sets: I

- *Sets*-based or concrete categories (i.e., with an underlying-sets functor) have objects as structured sets and morphisms as set functions that preserve (or reflect) that structure.
- To illustrate extending the analysis to such categories, we consider an example broached by Marquis.
- Marquis considers a category $\mathcal{C}$ with finite products, finite coproducts, and a null object (an object that is both initial and terminal). Then a canonical morphism can be constructed from the coproduct of two objects to the products of the objects.

Extending the analysis to categories based on Sets: II

- The simplest *Sets*-based category with those properties is the category $Sets_*$ of pointed sets whose objects are sets X with a designated element or basepoint $1 \xrightarrow{x_0} X$ and whose morphisms are set functions preserving the basepoints:

$$\begin{array}{ccc} 1 & & \\ \downarrow^{x_0} & \searrow^{y_0} & \\ X & \longrightarrow & Y \end{array}.$$

- Hence $Sets_*$ can also be seen as the slice category $1/Sets$ of *Sets* under 1.

Extending the analysis to categories based on Sets: III

- The maps that define the structure of structured sets (e.g., $1 \xrightarrow{x_0} X$) in a *Sets*-based category are considered canonical by definition so they may be composed with the logically-defined canonical maps to yield other canonical maps.
- The null object in $Sets_*$ is 'the' one-point set $1 \xrightarrow{id.} 1$.
  - The refinement relation $\mathbf{0}_X \precsim \mathbf{1}_X$ induces the unique map $X \rightarrow 1$ that makes 1 the terminal object in *Sets*. And since $1 \xrightarrow{x_0} X \rightarrow 1 = 1 \xrightarrow{id.} 1$, it is also the terminal object in $Sets_*$.
  - The basepoint in (Y, y_0) is given by the structurally canonical map $1 \xrightarrow{y_0} Y$ and since $1 \xrightarrow{id.} 1 \xrightarrow{y_0} Y = 1 \xrightarrow{y_0} Y$, that map $1 \xrightarrow{y_0} Y$ is the unique map that makes 1 also the initial object in $Sets_*$.

Extending the analysis to categories based on Sets: IV

  - Hence in *Sets*$_*$, there is always a canonical map formed by the composition: $X \to 1 \to Y = X \to Y$ (called the *zero arrow*).
- To build up the E&D analysis of the canonical morphism $X \sqcup_* Y \longrightarrow X \times_* Y$ from the coproduct to the product in *Sets*$_*$, we begin with the construction of the coproduct $X \sqcup_* Y$ which is just the pushout in *Sets* of the two canonical basepoint maps:

$$\begin{tikzcd}
1 \arrow[r, "x_0"] \arrow[d, "y_0"'] \arrow[dr] & X \arrow[r, equal] \arrow[d, "can.."] & X \arrow[dd, "h"] \\
Y \arrow[r, "can."] \arrow[d, equal] & X \sqcup_* Y = X \sqcup Y / \sim \arrow[dr, "\exists! \; h_*"] & \\
Y \arrow[rr, "h'"] & & U
\end{tikzcd}$$

Extending the analysis to categories based on Sets: V

- The only points identified in $X \sqcup Y/\sim$ are the basepoints. Since the maps $h : X \to U$ and $h' : Y \to U$ are assumed to be maps in $Sets_*$, the factor map $h_* : X \sqcup Y/\sim \to U$ from the pushout in $Sets$ will also preserve basepoints so $X \sqcup Y/\sim = X \sqcup_* Y$ is the coproduct in $Sets_*$.
- Similarly, the product $X \times_* Y$ in $Sets_*$ is just the product $X \times Y$ in $Sets$ with $\langle x_0, y_0 \rangle$ as the basepoint.
- Using the UMP of the product $X \times_* Y$ in $Sets_*$, the two $Sets_*$ maps $1_X : X \to X$ and the canonical $X \to 1 \to Y$, we have the unique canonical factor map $X \to X \times_* Y$ in $Sets_*$ and similarly for $Y \to X \times_* Y$ in $Sets_*$.

Extending the analysis to categories based on Sets: VI

- Then we put all the canonical maps together and use the UMP for the coproduct in $Sets_*$ to construct the desired canonical map: $X \sqcup_* Y \to X \times_* Y$ in $Sets_*$.

$$\begin{array}{ccccc} X & \xrightarrow{can.} & X \sqcup_* Y & \xleftarrow{can.} & Y \\ & \searrow^{can.} & \exists! \downarrow can. & can. \swarrow & \\ & & X \times_* Y & & \end{array}$$

Coproduct diagram in $Sets_*$

Extending the analysis to categories based on Sets: VII

- This example shows how in a *Sets*-based category like $Sets_*$, the given canonical maps for the structured sets (i.e., the basepoint maps $1 \xrightarrow{x_0} X$) are combined with the canonical maps defined by the E&D analysis in *Sets* to give the canonical morphisms in the *Sets*-based category.
- This is abstracted in abstract category theory (where there is no underlying structure of sets), as in the case of a category $\mathcal{C}$ which is assumed to have finite products, finite coproducts, and a null object. The 'atomic' canonical morphisms are all given as part of the assumed UMPs for products, coproducts, and the null object which are then composed to define other 'molecular' canonical morphisms.

Logical theory of canonicity: I

- "Logical" refers to the two dual mathematical logics: the Boolean logic of subsets and the logic of partitions.
- Note that the logic of subsets and the logic of partitions have equal intertwining roles in the whole analysis.
- Normally, we might say that "subsets" and "partitions" are category-theoretic duals, but we have tried to show a more fundamental analysis based on "elements & distinctions" or "its & dits" that are the building blocks of subsets and partitions and that underlie the duality in *Sets*.
- The dual interplay of elements and distinctions explains morphisms, duality, canonicity, and universal constructions in *Sets*, which generalizes to other *Sets*-based concrete categories and which is abstracted in abstract category theory.

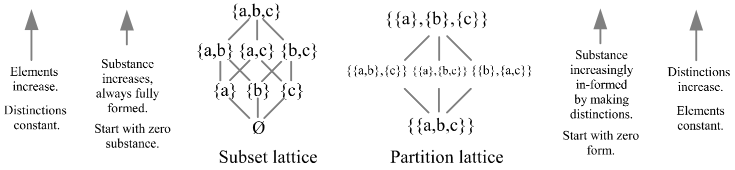
Logical theory of canonicity: II

- Our focus here is the E&D treatment of canonicity.
- Each construction starts with certain data.
- When that data is sufficient to define inclusions in an associated subset lattice or refinements in an associated partition lattice, then the resulting injections and surjections (and their compositions) are canonical.
- That is the logical theory of canonicity.

Appendix: Speculative concluding remarks: I

- We shown how the dual concepts of elements & distinctions can be used to account for the notion of morphism, duality, and the universal constructions in *Sets*–which are then abstracted in abstract category theory.
- Hence the E&D notions may have a broader philosophical significance.
- One possibility is they are respectively mathematical versions of the old metaphysical concepts of *matter* (or *substance*) and *form* (as in in-form-ation).

Appendix: Speculative concluding remarks: II



- At the bottom of the Boolean subset lattice is the empty set $\emptyset$ which represents no substance or matter (no 'its').
- As one moves up the lattice, new elements of substance are created that are always fully formed until finally one reaches the top, the universe U.

Appendix: Speculative concluding remarks: III

- At the bottom of the partition lattice is the indiscrete partition or "blob" $\mathbf{0} = \{U\}$ (where the universe set U makes one block) which represents all the substance or matter but with no distinctions to in-form the substance (no 'dits').
- As one moves up the lattice, no new substance is created but distinctions are created that in-form the indistinct elements as they become more and more distinct until finally one reaches the top, the discrete partition $\mathbf{1}$, where all the elements of U have been fully formed.
- Thus one ends up at the "same place" (universe U of fully distinguished elements) either way, but by two totally different but dual 'creation stories':

Appendix: Speculative concluding remarks: IV

- creating elements (as in creating fully-formed matter out of nothing) versus
- creating distinctions (as in starting with a totally undifferentiated matter and then in a 'big bang' start making distinctions, e.g., breaking symmetries, to give form to the matter).

- Moreover:
  - the quantitative increase in substance (normalized number of elements) moving up in the subset lattice is measured by logical (Laplacian) probability, and
  - the quantitative increase in form (normalized number of distinctions) moving up in the partition lattice is measured by logical information.