Automata and topological theories

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Motivation.

Why study topological quantum field theory (TQFT)? It computes topological invariants. It is also related to knot theory, 4-manifolds, algebraic topology, moduli spaces in algebraic geometry.

Category theory is a way of recognizing constructions that appears when you are talking about different kinds of mathematical objects that have something in common.

Example: cartesian product of abelian groups, cartesian product of manifolds, etc. Although the objects are different, the general notion of cartesian product is the same.

TQFT produces a tower of algebraic structures, each dimension related to the previous one by the process of *categorification*.

Remark: some higher dimensional theories exist but are not well-understood.

Background.

Definition: a *manifold* is a smooth, compact, (oriented) topological space.
Note: manifolds may have boundaries.
Definition: a *closed manifold* is a manifold without a boundary.
Examples:

- a sphere
- a donut
- a notebook paper (a 2-manifold with 1-dimensional boundary)
- the boundary of an d-manifold is an $(d-1)$-manifold.

Organize closed manifolds with a fixed dim as a category Cob(d):

- objects: closed $(d-1)$-manifolds,
- morphisms: $\mathrm{Hom}(M, N) \simeq \{B : \partial B \simeq \overline{M} \sqcup N\}/\,\mathrm{diffeom}$, where $\overline{M}$ is M in opposite orientation.

Two bordisms are the same in Cob(d) if they are diffeomorphic relative to their boundary, and composition is given by gluing the morphisms.

The definition of TQFT.

Definition

TQFT of dimension d is a symmetric, monoidal functor

$$Z : \mathrm{Cob}(d) \longrightarrow \mathrm{Vect}_{\mathbb{C}},$$

which preserves tensor products $\otimes$.

The $\otimes$ in $\mathrm{Cob}(d)$ is given by disjoint union of manifolds while $\otimes$ in $\mathrm{Vect}_{\mathbb{C}}$ is given by the tensor product of vector spaces:

$$Z(M \sqcup N) = Z(M) \otimes Z(N), \quad Z(\varnothing) \simeq \mathbb{C},$$

where $\mathbb{C}$ is a unit with respect to the tensor product on $\mathbb{C}$-vector spaces.

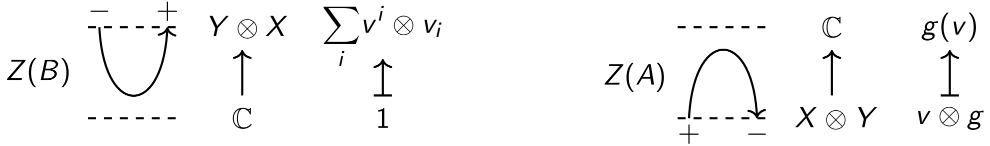
Example: Cob(1). Let $d = 1$.

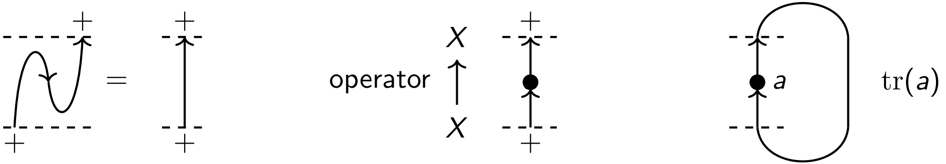
Objects are 0-dimensional manifolds with orientation: $\overset{+}{\bullet}$, $\overset{-}{\bullet}$.

$Z\left(\overset{+}{\bullet}\right) = X$ finite dimensional $\mathbb{C}$-vector space, where $\overset{+}{\bullet}$ has positive orientation.

$Z\left(\overset{-}{\bullet}\right) = Y$ finite dimensional $\mathbb{C}$-vector space, where $\overset{-}{\bullet}$ has negative orientation.

Oriented 1D TQFT is a pair X, Y satisfying:





The maps $Z(A)$ and $Z(B)$ exhibit that X and Y must be duals of each other: $Y \simeq X^\vee$.

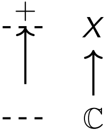
Conclusion: 1-dimensional TQFT is determined by what it does to a single point, i.e., it is determined by a single vector space.

One can evaluate field theory on other manifolds to get invariants. For example, evaluate field theory on a circle with no defects to get the only invariant on a complex vector space, which is its dimension:

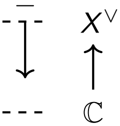
a complex vector space is determined up to isomorphism by its dimension.

Inner boundary points and defects.

Add inner boundary points.

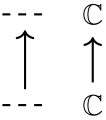


$v \in X$ vector



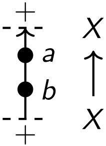
$f : X \to \mathbb{C}$ linear map

floating interval

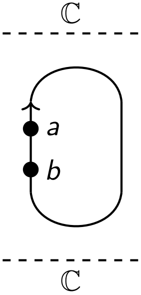


$f(v) \in \mathbb{C}$

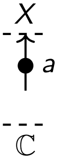
Add defects.



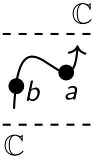
$a, b : X \to X$ linear operators
(no relations)



$= \mathrm{tr}(ab)$



$= av \in X$



$= f(abv)$

Left: circle with a sequence of defects computes the trace of the corresponding product of operators.

Center: a defect near an endpoint applies the operator to the vector associated to the endpoint for "in" oriented endpoints.

Right: for "out" endpoint, the operator acts on a functional f.

Evaluation on closed diagrams.

Assume that a more general evaluation is given, but only on floating (closed) diagrams, intervals and circles with defects.

A floating interval with defects $a_1 \cdots a_n$ evaluates to $\alpha_{\mathsf{I}}(a_1 \cdots a_n) \in \mathbb{C}$.

A circle with defects $b_1 \cdots b_m$ evaluates to $\alpha_\circ(b_1 \cdots b_m) \in \mathbb{C}$.

$\Rightarrow \alpha_{\mathsf{I}}, \alpha_\circ$ are functions from words (resp. circular words) to $\mathbb{C}$.

Assume letters (defects) belong to a finite alphabet Σ.

Σ^*: the set (monoid) of all words in Σ. Then

$$\alpha_{\mathsf{I}} : \Sigma^* \longrightarrow \mathbb{C}, \quad \alpha_\circ : \Sigma^*/_\sim \longrightarrow \mathbb{C}$$

are two functions, where $\sim$ is the equivalence relation on words: $\omega_1\omega_2 \sim \omega_2\omega_1$ for words ω_1, ω_2.

Given such a pair $\alpha = (\alpha_{\mathsf{I}}, \alpha_{\circ})$, build a generalized 1D topological theory (this is weaker than a TQFT).

Extend evaluation α to unions of decorated circles and floating intervals via multiplicativity condition.

In a universal construction of topological theories, one starts with a multiplicative evaluation of closed objects (such as closed d-manifolds) and builds a vector space for each $(d-1)$-manifold N via a linear combination of d-manifolds M with boundary N, $\partial M \cong N$. A linear combination $\sum_i \lambda_i M_i = 0$ with each $\partial M_i \cong N$ if for any M with $\partial M \cong N$, the evaluation

$$\sum_i \lambda_i \alpha(\overline{M} \cup_N M_i) = 0.$$

Add defects to manifolds $\Rightarrow$ one-dimensional $(d=1)$ case becomes nontrivial.

Changing from ground field, such as $\mathbb{C}$, to a semiring (for example, Boolean semiring $\mathbb{B}$) further adds complexity to the theory and surprisingly relates it to regular languages and automata.

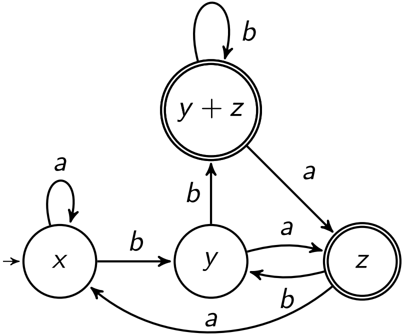
We will now discuss this nonlinear and more complicated case.

$\mathbb{B} = \{0, 1 : 1 + 1 = 1\}$ Boolean semiring.

Σ: alphabet (a finite set of letters). Σ^*: free monoid on the letters Σ.

Example: $\Sigma = \{a, b\}$. Words aaa, $ababbba$, $bbaaab$, etc. Empty word $\emptyset$ is unit element.

FSA (Finite State Automaton): words in Σ are inputs; finitely many states Q and transitions between the states $\Sigma \times Q \to Q$ according to the letters read. Has initial (starting) state q_{in} and terminating (accepting) states Q_t. Example:



Language $L = (a + b)^* b(a + b)$.
Second from last letter is b. Four states.
Initial state given by the empty word $q_{\text{in}} = x$.
Accepting states $Q_t = \{z, y + z\}$.
The states z and $y + z$ correspond to the words $(a + b)^* ba$ and $(a + b)^* bb$, respectively.
Notation $y + z$ comes from relation to $\mathbb{B}$-modules.

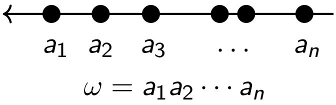
Regular language: one recognized by an FSA.

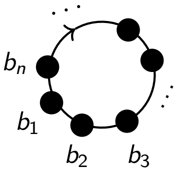
A word can be viewed as an interval with dots (defects) labelled by letters of the language L_{I}. Reading a sequence along oriented interval gives a word $\omega = a_1 a_2 \cdots a_n$.

Evaluation $\alpha_{\mathsf{I}} : \Sigma^* \longrightarrow \mathbb{B}$ of decorated intervals is the same as a (floating) interval language L_{I}: $\omega \in L_{\mathsf{I}} \Leftrightarrow \alpha_{\mathsf{I}}(\omega) = 1$.

Add a circular language $L_\circ$ (for words on a circle $\omega_1\omega_2 \in L_\circ \Leftrightarrow \omega_2\omega_1 \in L_\circ$).

With pair $L = (L_{\mathsf{I}}, L_\circ)$, associate a $\mathbb{B}$-valued multiplicative evaluation α of decorated 1-manifolds (defects labelled by letters in Σ).





$$\alpha \ : \ \text{closed 1-dimensional manifolds} \longrightarrow \mathbb{B} \quad \text{which satisfies}$$

$$\begin{aligned} \alpha(M_1 \sqcup M_2) &= \alpha(M_1)\alpha(M_2), \\ \alpha(\emptyset_1) &= 1 \quad \text{since } m \text{ is multiplicative}, \\ \alpha(M_1) &= \alpha(M_2) \ \text{ if } M_1 \cong M_2. \end{aligned}$$

View interval as a "closed" 1-manifold.
$\alpha = (\alpha_{\mathsf{I}}, \alpha_\circ)$ is determined by its values $\alpha_{\mathsf{I}}(\omega)$ on decorated floating intervals and values $\alpha_\circ(\omega)$ on decorated circles:

$$\alpha_{\mathsf{I}}(\omega) = 1 \Leftrightarrow \omega \in L_{\mathsf{I}} \quad \text{and} \quad \alpha_\circ(\omega) = 1 \Leftrightarrow \omega \in L_\circ.$$

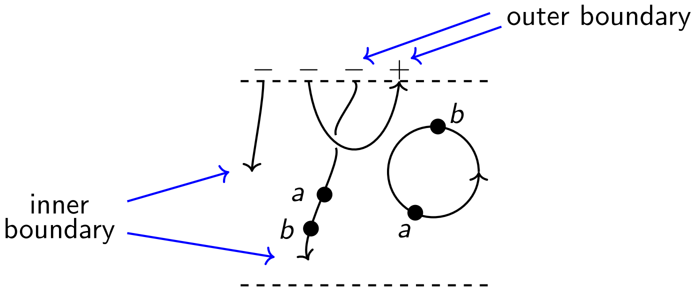
Universal construction starts with a (multiplicative) evaluation of closed n-dimensional objects and produces state spaces for $(n-1)$-dimensional objects and maps for n-cobordisms between these objects.

Use universal construction to define state spaces of oriented 0-dimensional manifolds (sign sequences $\varepsilon = (- - -+)$, for example).

Sign sequence: $\varepsilon = (- - -+)$. Sign sequences are objects of our category of 1-dim cobordisms with 0-dim defects in Σ.

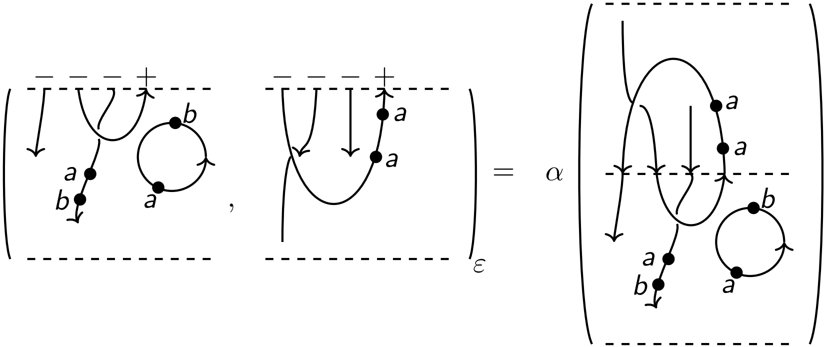
From α, one can define state spaces $A(\varepsilon)$ for 0-dimensional objects ε, by starting with a free $\mathbb{B}$-semimodule $\mathrm{Fr}(\varepsilon)$ with a basis $\{[M]\}_{\partial M \cong \varepsilon}$ given by formal symbols $[M]$ of all 1-dimensional objects M which have ε as outer boundary (with a fixed diffeomorphism $\partial M \cong \varepsilon$).

A state in the state space $A(\varepsilon)$:



On $\mathrm{Fr}(\varepsilon)$, introduce a bilinear pairing $(\ ,\ )_\varepsilon$ given on basis elements $[M_1], [M_2]$ with $\partial M_1 \cong \varepsilon \cong \partial M_2$ by coupling M_1, M_2 along the boundary and evaluating the resulting closed object $M_1 \cup_\varepsilon M_2$ via α:

$$([M_1],[M_2])_\varepsilon \ := \alpha(M_1 \cup_\varepsilon \overline{M_2}).$$



Note that $A(+) \cong A(-)^* = \mathrm{Hom}(A(-), \mathbb{B})$ via $\omega \mapsto (\omega' \mapsto \alpha(\omega'\omega)) \in \mathbb{B}$.

Now define the state space $A(\varepsilon)$ as the quotient of $\mathrm{Fr}(\varepsilon)$ by an equivalence relation,

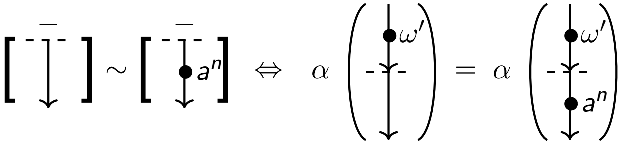
$$A(\varepsilon) \;:=\; \mathrm{Fr}(\varepsilon)/\sim,$$

where $\sum_i [M_i] \sim \sum_j [M_j']$ if for any M with $\partial M = \varepsilon$,

$$\sum_i \alpha(M_i \cup_\varepsilon \overline{M}) = \sum_j \alpha(M_j' \cup_\varepsilon \overline{M}) \;\in\; \mathbb{B} = \{0, 1 : 1 + 1 = 1\}.$$

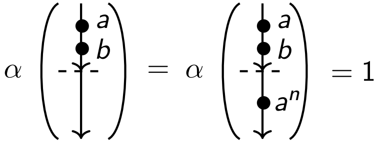
State space $A(\varepsilon)$ is spanned by $\mathbb{B}$-linear combinations of 1-manifolds M with $\partial M \cong \varepsilon$, modulo relations: two linear combinations are equal if for any way to close them up and evaluate using α, the result is the same.

One of the relations for the language $L_{\mathrm{I}} = (a+b)^* b (a+b)$:

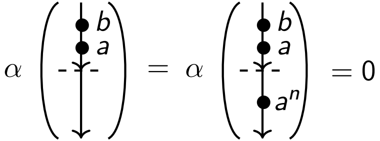


for any $\omega' \in \Sigma^*$.

If $\omega' = ba$, then



If $\omega' = ab$, then



State spaces $A(-), A(+)$ depend only on the interval language L_{I}, not on the circular language $L_{\circ}$ (spaces $A(+-)$, etc. depend on both).

An evaluation table of the language $L = (a+b)^* b(a+b)$ to compute the bilinear form on our spanning sets for $A(+)$ and $A(-)$ with values in $\mathbb{B}$. The matrix is not symmetric.

	spanning elmt	x	x	y	x	z	y	$y+z$
spanning elmt		↓	↓ a	↓ b	↓ a a	↓ a b	↓ b a	↓ b b
x'	↑	0	0	0	0	1	0	1
y'	↑ a	0	0	1	0	0	1	1
y'	↑ b	0	0	1	0	0	1	1
0	↑ a a	0	0	0	0	0	0	0
0	↑ a b	0	0	0	0	0	0	0
z'	↑ b a	1	1	1	1	1	1	1
z'	↑ b b	1	1	1	1	1	1	1

Defining relations:

$$x + y = y$$

$$x + z = z$$

$$A(-) = \frac{\mathbb{B}x \oplus \mathbb{B}y \oplus \mathbb{B}z}{\langle x+y=y, x+z=z \rangle}$$

Consists of 5 elements:

$$\{0, x, y, z, y+z\},$$

with x, y, z irreducible.

State space of $A(+-)$ is spanned by:

A 1-manifold M with $\partial M = \varepsilon' \sqcup -\varepsilon$ induces a map $A(\varepsilon) \longrightarrow A(\varepsilon')$ by concatenation.

Get a functor from category of Σ-decorated oriented 1-dim cobordisms to $\mathbb{B}$-semimodules. No subtraction in $\mathbb{B}$-semimodules; can add only.

A $\mathbb{B}$-semimodule V is a commutative idempotented monoid under addition:

$$x + x = x \text{ for } x \in V \text{ since } 1 + 1 = 1. \text{ Also } 0 + x = x,$$
$$x + y = y + x, \qquad (x + y) + z = x + (y + z).$$

Such V correspond to sup-semilattices, with join (least upper bound) $x \vee y := x + y$, and $x \leqslant y$ iff $x + y = y$.

0 is the minimal element, i.e., $0 \leqslant x$ for any x.

Any finite sup-semilattice is a finite lattice, with meet $x \wedge y := \sum_{z \leqslant x, y} z$ and $1 = \sum_{z \in V} z$.

We mostly use $\mathbb{B}$-semimodule structure (join, not meet).

$$\mathbb{B}\text{-semimodules} \Leftrightarrow \text{comm. idemp. monoids} \Leftrightarrow \text{sup-semilattices (with 0)}$$

$$\text{finite (sup)-semilattices} \Leftrightarrow \text{finite lattices}$$

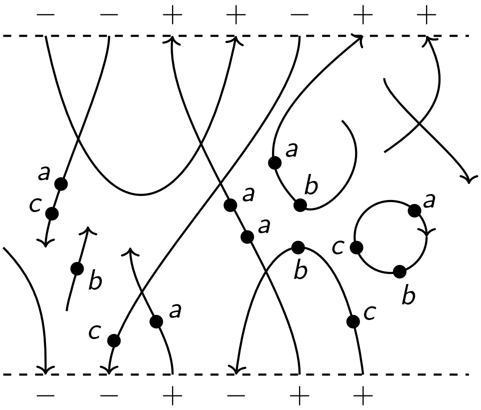
$\mathbb{B}$-semimodules constitute a category; morphisms are semimodule homomorphisms $f : V \longrightarrow W$, $f(0) = 0$, $f(x+y) = f(x) + f(y)$.

Hom(V, W) is a $\mathbb{B}$-semimodule (category $\mathbb{B}-$mod has internal homs). But $\mathbb{B}-$mod is not a rigid category (cannot "bend" objects and morphisms).

Subcategory of finite projective $\mathbb{B}$-semimodules (finite distributive (semi)lattices) is rigid.

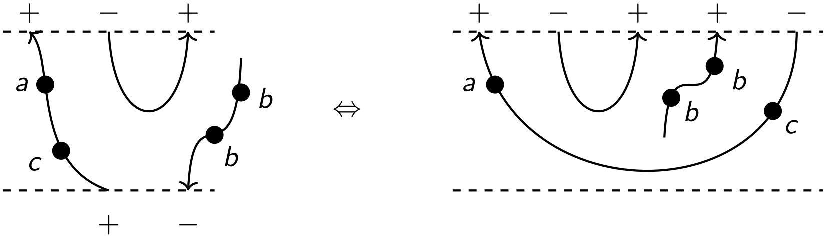
Categories of cobordisms in the universal construction that we build from evaluations are rigid.

Any cobordism C between $\varepsilon, \varepsilon'$ induces a semimodule homomorphism $A(\varepsilon) \to A(\varepsilon')$ of concatenation with C:



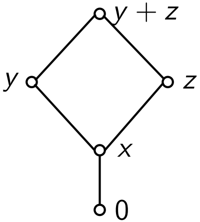
A cobordism from $(- - + - ++)$ to $(- - + + - + +)$.

A cobordism from ε to ε' can be viewed as an element in the state space $A(\varepsilon' \sqcup -\varepsilon)$, i.e., a cobordism $C : \varepsilon = (+-) \to \varepsilon' = (+ - +)$ corresponds to a state in the state space $A(+ - + + -)$:

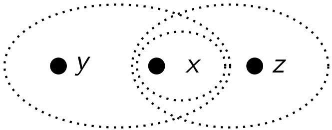


Recall the language $L = (a+b)^* b(a+b)$. The module $A(-)$ is spanned by x, y, z, and has relations $x + y = y$ and $x + z = z$. This module is not free. We'll encounter its free cover later in the construction of minimal NFA (nondeterministic FA) for L.

The semimodule consists of 5 elements: $\{0, x, y, z, y+z\}$. The lattice corresponding to this language is:



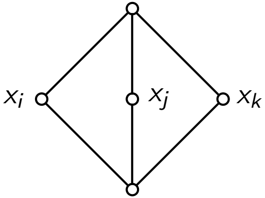
The finite topological space associated to this example:



Lattices that come from finite topological spaces are distributive.

If a lattice contains either as a sublattice,

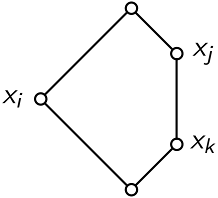
$$x_i + x_j = x_i + x_k = x_j + x_k$$



$$x_i \cap x_j = x_i \cap x_k = x_j \cap x_k$$

$$x_i \cap x_j + x_j = x_j$$

$$x_i \cap x_j < x_i, x_j, x_k$$



$$x_i + x_j = x_i + x_k \text{ for } j < k$$

$$x_j + x_k = x_j \text{ for } j < k$$

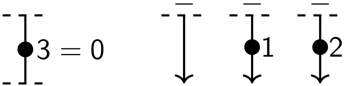
$$x_i \cap x_k < x_i, x_k$$

$$x_i, x_j < x_i + x_j$$

then the lattice is not distributive.

In such a case, there is no finite topological space associated to the language.

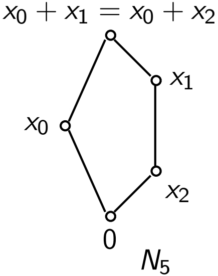
Example: for the language $L_\mathsf{I} = \{a, a^2\}$, lattices $A(-), A(+)$ are not distributive.



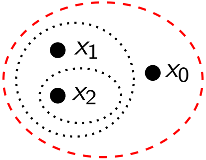
x_0	0	1	1
x_1	1	1	0
x_2	1	0	0

$$x_1 + x_2 = x_1$$

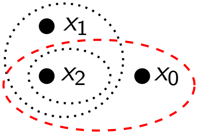
$$x_0 + x_1 = x_0 + x_2$$



For the language $L_{\text{I}} = \{a, a^2\}$, how should we draw the finite topological space associated to L_{I}?



But $x_0 \neq x_0 + x_1$. So the open set containing x_0 cannot be the entire space.



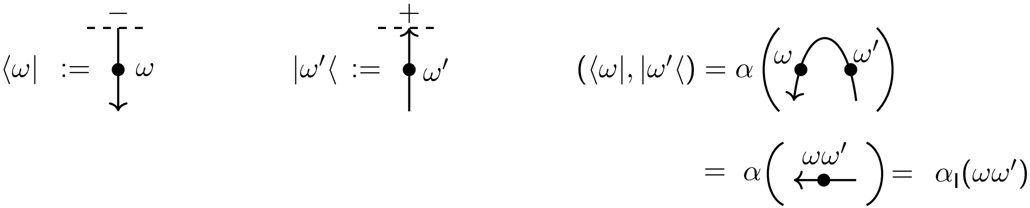
But since $x_0 \neq x_0 + x_2$, this finite topological space does not correspond to L_{I} as well.

Theorem. Languages $L_{\mathsf{I}}, L_{\circ}$ are regular $\Leftrightarrow$ the state space $A(\varepsilon)$ is a finite $\mathbb{B}$-semimodule for all sequences ε.

Get a $\mathbb{B}$-valued topological theory with finite hom spaces for any such pair of languages.

To recover minimal automaton for L_{I}, consider the state space $A(-)$. It consists of $\mathbb{B}$-linear combinations of diagrams below on the left, modulo equivalence relations coming from the pairing

$$A(-) \times A(+) \longrightarrow \mathbb{B}.$$



How do we build the minimal deterministic FSA and nondeterministic FSA for L_{I} from $A(-)$?

Free monoid Σ^* generated by Σ (monoid of words) acts on $A(-)$, by composing with dots at the end of the strand.

State space $A(-)$ contains the subset $Q^- = \{\langle\omega|\}$ of *pure* states. Q^- is then the set of states of the minimal *deterministic* FSA for L_{I}. Action of Σ comes from restriction of its action on $A(-)$ (action by concatenation with dots at the top).

Initial state $q_{\mathrm{in}} = \langle\emptyset|$. A state $\langle\omega|$ is accepting iff $\alpha_{\mathrm{I}}(\omega) = 1$. *Nondeterministic* FSA for L_{I} come from coverings of $A(-)$ by free $\mathbb{B}$-modules with lifted action of Σ and unit, trace α maps.

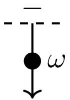
$\widetilde{m}_a \circlearrowright \mathbb{B}^J$ free semimodule cover; minimal NFA for L_{I}, where $J = \mathrm{irr}(A(-))$ (irreducible if $x \neq y + z$, where $y \neq x$, $z \neq x$)

$\downarrow$

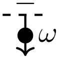
$m_a \circlearrowright A(-)$ state space of 0-manifold $-$

$\uparrow$

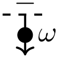
$m_a \circlearrowright Q_-$ minimal DFA for L_{I}



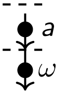
Every word gives a diagram in $A(-)$.

Start with a state 

and take images of all $\omega \in A(-)$ under the action by Σ^*, i.e.,



$$= \omega = \langle \omega | \in A(-) \mapsto$$

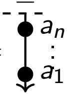


$$=$$

$$\omega a \in A(-) \mapsto \begin{cases} 1 & \text{if } \omega a \in L_{\mathsf{I}}, \\ 0 & \text{if } \omega a \notin L_{\mathsf{I}}. \end{cases}$$

$$q_{\mathrm{in}} =$$

$$= \langle \varnothing | \overset{a_1}{\mapsto} \langle a_1 | \overset{a_2}{\mapsto} \langle a_1 a_2 | \mapsto \ldots \overset{a_n}{\mapsto} \langle a_1 a_2 \cdots a_n | =$$

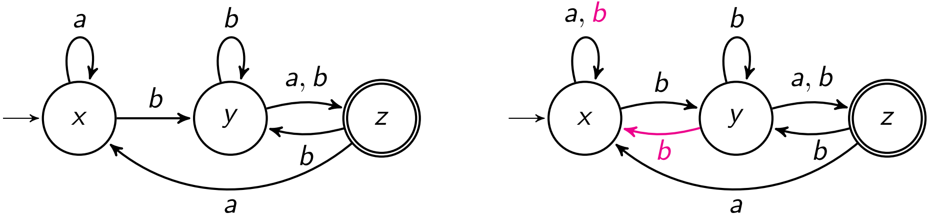


$$\mapsto \begin{cases} 1 & \text{if } a_1 \cdots a_n \in L_{\mathsf{I}}, \\ 0 & \text{if } a_1 \cdots a_n \notin L_{\mathsf{I}}. \end{cases}$$

In general, there could be more than 1 minimal NFA.

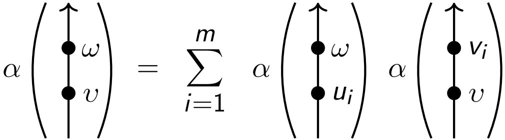
Two minimal nondeterministic automata on 3 states that accept the language $L = (a+b)^* b(a+b)$.

The second automaton has an additional b arrow from y to x and an additional b loop at x.



Multiple minimal NFA for L appear due to several ways of lifting action of Σ^* from $A(-)$ to $\mathbb{B}^J$.

Some regular languages allow decomposition of identity

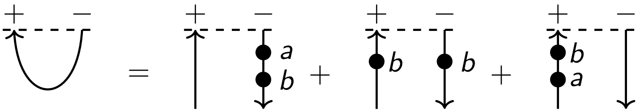


for some set of pairs of words (u_i, v_i), $1 \leqslant i \leqslant m$.

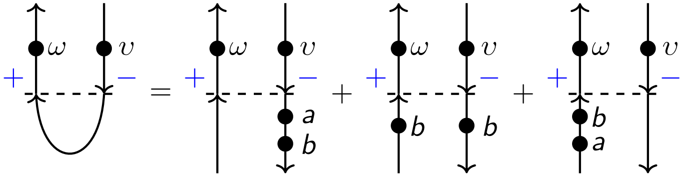
That is, for any $\omega, \upsilon \in \Sigma^*$,

$$\alpha_I(\omega\upsilon) = \sum_{i=1}^{m} \alpha_I(\omega u_i)\, \alpha_I(v_i \upsilon).$$

Returning to our example $L = (a+b)^*b(a+b)$,

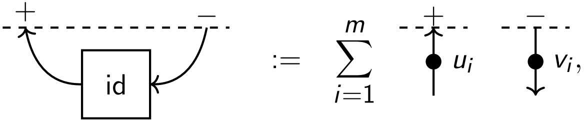


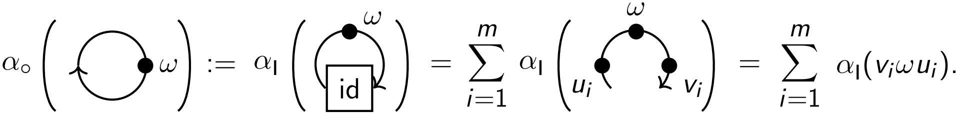
So



$$\alpha_{\mathsf{I}}(\omega \upsilon) = \alpha_{\mathsf{I}}(\omega)\,\alpha_{\mathsf{I}}(ba\upsilon) + \alpha_{\mathsf{I}}(\omega b)\,\alpha_{\mathsf{I}}(b\upsilon) + \alpha_{\mathsf{I}}(\omega ba)\,\alpha_{\mathsf{I}}(\upsilon).$$

For L_{I} with a decomposition of the identity, there is a unique associated circular language such that the decomposition still holds:





This gives a $\mathbb{B}$-valued TQFT: $A(\varepsilon)$ is the tensor product of $A(+), A(-)$ for the sequence of signs in ε.

For example, $A(++-) \cong A(+) \otimes A(+) \otimes A(-)$.

This is a TQFT for oriented 1-manifolds with 0-dimensional Σ-labelled defects, valued in the Boolean semiring $\mathbb{B}$.

Proposition. A regular language L has a decomposition of the identity if and only if $A(-)$ is a projective $\mathbb{B}$-semimodule (equivalently, a distributive lattice).

A finite semimodule P is projective if it's a retract of a free semimodule:

$$P \xrightarrow{\iota} \mathbb{B}^n \xrightarrow{p} P, \quad p\iota = \mathrm{id}_P.$$

Note that $\iota \circ p$ is an idempotent.

Such semimodules correspond to finite topological spaces X, with elements of the semimodule given by open subsets $U \subset X$ and $U + V := U \cup V$.

Summary.

A pair $\alpha = (\alpha_{\mathsf{I}}, \alpha_{\circ})$ gives rise to a Boolean topological theory (state spaces are finite) iff $\alpha_{\mathsf{I}}, \alpha_{\circ}$ are regular languages.

Such a theory is a weakly monoidal functor from the category of oriented 1D cobordisms with Σ-defects to the category of finite (semi)modules over $\mathbb{B}$.

State space $A(-)$ is determined by α_{I} only.

If $A(-)$ is a projective $\mathbb{B}$-semimodule (comes from a finite topological space), there is a unique circular language $\alpha_{\circ}$ making α into a Boolean 1D TQFT with defects (maps $A(\varepsilon) \otimes A(\varepsilon') \longrightarrow A(\varepsilon\varepsilon')$ are isomorphisms of state spaces).

Work in progress.

1. Distributivity of $A(-)$ is a subtle property of a regular language α_I, even for $\Sigma = \{a\}$ (single letter). Study distributivity of regular languages (joint with R. Kaldawy, M. Khovanov, Z. Lihn).
2. Any NFA for α_I gives rise to a circular language (via all cycles in the NFA) and a 1D TQFT with defects, even when $A(-)$ is not projective. Study these TQFTs.
3. Allow defects to accumulate towards inner endpoints. Evaluation of infinite words. Resulting topological theories relate to sofic systems and symbolic dynamics (joint with M. Khovanov, P. Gustafson).
4. Automata with boundary. Boolean evaluations beyond automata.
5. Boolean two-dimensional topological theories and TQFTs. Ultimately hope to study these topological theories in dimension three as well.

Thank you!