# The core groupoid can suffice

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Zooming from Terramerragal country

**Dedicated to the memory of Marta Bunge,**

**colleague and dear family friend**

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# The general problem

- Let $R$ be a commutative ring. Let $\mathscr{V} = \mathrm{Ab}^R$ be the monoidal category of $R$-modules. For any category $\mathscr{F}$, let's call the functor category $[\mathscr{F}, \mathscr{V}]$ the *category of $R$-linear representations of $\mathscr{F}$*.

- I am interested in categories $\mathscr{F}$ for which there is a groupoid $\mathscr{G}$ such that the categories $[\mathscr{F}, \mathscr{V}]$ and $[\mathscr{G}, \mathscr{V}]$ of representations are equivalent.

- In particular, $\mathscr{G}$ could be the core groupoid $\mathscr{F}_{\mathrm{inv}}$ of $\mathscr{F}$; that is, the subcategory with the same objects and with only the invertible morphisms.

- We could freely split some idempotents in $\mathscr{F}$ without changing the representations. However, adding the new objects may change the core groupoid.

# Linearizing

- Every category $\mathscr{F}$ gives rise to a $\mathscr{V}$-category (that is, an $R$-linear category), denoted $R\mathscr{F}$, with the same objects and with hom $R$-module $R\mathscr{F}(A, B)$ free on the homset $\mathscr{F}(A, B)$.
- Indeed, $R\mathscr{F}$ is the free $\mathscr{V}$-category on $\mathscr{F}$ so that the $\mathscr{V}$-functor category $[R\mathscr{F}, \mathscr{V}]$ is isomorphic to the ordinary functor category $[\mathscr{F}, \mathscr{V}]$ with the pointwise $R$-linear structure.
- In these terms, we are interested in when $R\mathscr{F}$ and $R\mathscr{G}$ are Morita equivalent $\mathscr{V}$-categories so that a groupoid suffices for all linear representations of $\mathscr{F}$.
- Lack-St "Combinatorial categorical equivalences of Dold-Kan type" showed that proving a Dold-Kan-type equivalence of categories reduced to solving a "core groupoid suffices" problem.

# An example without details

- Indeed the original Dold-Kan equivalence between simplicial $R$-modules and chain complexes of $R$-modules reduces to the "core groupoid suffices" problem for $\mathscr{F} = \Delta_\perp$.

- Here $\Delta_\perp$ is the category of finite non-empty ordinals

$$\mathbf{a} = \{0, 1, \ldots, a-1\}$$

  and order-and-first-element preserving functions $\xi : \mathbf{a} \to \mathbf{b}$.

- As the only invertible morphisms in $\Delta_\perp$ are identities, the core groupoid in this case is discrete with countably many objects; so the category of representations is the product $\mathscr{V} \times \mathscr{V} \times \ldots$ of countably many copies of $\mathscr{V}$, that is, the category of objects of $\mathscr{V}$ graded by the positive integers.

# A baby example with details, page 1

- Let $C = \mathbf{3} = \{0, 1, 2\} \in \Delta_\perp$. The monoid $\Delta_\perp(C) = \Delta_\perp(\mathbf{3}, \mathbf{3})$ has six elements, namely, the functions

$$\iota = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \qquad \sigma_0 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \sigma_1 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \qquad \upsilon = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \zeta = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- A presentation of this monoid is provided by the generators $\sigma_0$, $\sigma_1$ and $\tau$ subject to the relations that the generators are idempotents and

$$\sigma_0\sigma_1\sigma_0 = \sigma_0\sigma_1 = \sigma_1\sigma_0\sigma_1$$

$$\tau\sigma_0 = \sigma_0 \ , \ \tau\sigma_1 = \tau = \sigma_0\tau \ , \ \sigma_1\tau = \sigma_1 \ .$$

# A baby example with details, page 2

- The only invertible element in $\Delta_\perp(C)$ is the identity $\iota$.
- The core groupoid of $\Delta_\perp(C)$ is a discrete category with one object. Clearly $R\Delta_\perp(C)$ is not Morita equivalent to $R1$.
- Let $\Delta'_\perp(C)$ denote the five-element semigroup obtained by omitting $\iota$.
- **NB** $R\Delta'_\perp(C)$ (with multiplication coming from the semigroup) is actually an $R$-algebra with identity element $\ell_C = \sigma_0 + \sigma_1 - \upsilon$.

# A baby example with details, page 3

- Now in $R\Delta_\perp(C)$ we have a complete list of orthogonal idempotents:

$$e_0 = \sigma_0\sigma_1 \ , \ e_1 = (1-\sigma_0)\sigma_1 \ ,$$
$$e_2 = (1-\sigma_1)\sigma_0 \ , \ e_3 = (1-\sigma_1)(1-\sigma_0) \ ,$$

  so that, in the "Cauchy completion" $\mathscr{Q}R\Delta_\perp(C)$ (freely split idempotents and freely adjoin direct sums), we have

$$C \cong E_0 \oplus E_1 \oplus E_2 \oplus E_3$$

  where $E_i$ is the object obtained in splitting the idempotent $e_i$.

- If we put $\gamma = \sigma_1\sigma_0 - \sigma_0\sigma_1$ and $\delta = \tau - \sigma_1$, using the relations in the presentation, we obtain the equations

$$e_1\gamma = \gamma = \gamma e_2 \ , \ \delta e_1 = \delta = e_2\delta \ , \ \gamma\delta = e_1 \ , \ \delta\gamma = e_2$$

  yielding $E_1 \cong E_2$. It follows that we have an isomorphism of the form

$$C \cong D \oplus E \oplus E \oplus F \ .$$

# A baby example with details, page 4

- Transporting the endomorphisms $\sigma_0$, $\sigma_1$, $\tau$ on $C$ across this isomorphism yields the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Any endomorphism $f$ of $D \oplus E \oplus E \oplus F$ which commutes with these three matrices must be of the form $f = f_0 \oplus f_1 \oplus f_1 \oplus f_2$.
- It follows that $R\Delta_\perp(C)$ is $\mathscr{V}$-Morita equivalent to $R3$ where 3 is the discrete category with three objects. In other words, we have an equivalence of $R$-linear categories

$$[\Delta_\perp(C), \mathscr{V}] \cong \mathscr{V} \times \mathscr{V} \times \mathscr{V} \ .$$

# Down to business: Factorization systems

1. Let $\mathscr{F}$ be a category with a factorization system $(\mathscr{E}, \mathscr{M})$. That is, $\mathscr{E}$ and $\mathscr{M}$ are sets of morphisms of $\mathscr{F}$ satisfying
   - FS0. for $w$ invertible, $mw \in \mathscr{M}$ if $m \in \mathscr{M}$, while $we \in \mathscr{E}$ if $e \in \mathscr{E}$;
   - FS1. if $mh = ke$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ then there exists a unique $\ell$ with $\ell e = h$ and $m\ell = k$;
   - FS2. every morphism $f$ factors as $f = me$ for some $m \in \mathscr{M}$ and $e \in \mathscr{E}$.

2. It follows that $\mathscr{E} \cap \mathscr{M}$ is the set of invertible morphisms and that $\mathscr{E}$ and $\mathscr{M}$ are both closed under composition in $\mathscr{F}$.

3. We identify $\mathscr{E}$, $\mathscr{M}$ and $\mathscr{G} := \mathscr{E} \cap \mathscr{M}$ with the subcategories of $\mathscr{F}$ containing all objects.

4. Write $\mathscr{M}'$, $\mathscr{E}'$, $\mathscr{G}'$ for the sets of morphisms of $\mathscr{F}$ **not** in $\mathscr{M}$, $\mathscr{E}$, $\mathscr{G}$, respectively.

# Proper factorization systems

The factorization system is called *proper* when

FSP. every member of $\mathscr{E}$ is an epimorphism and every member of $\mathscr{M}$ is a monomorphism.

If $(\mathscr{E}, \mathscr{M})$ is a proper factorization system on $\mathscr{F}$ then $(\mathscr{M}, \mathscr{E})$ is a proper factorization system on $\mathscr{F}^{\mathrm{op}}$.

## Proposition

*In a proper factorization system, if $hf \in \mathscr{M}$ then $f \in \mathscr{M}$.*
*Dually, if $fk \in \mathscr{E}$ then $f \in \mathscr{E}$.*

# The bicategory $\mathfrak{R} := \mathscr{V}\text{-Mod}$

Objects of $\mathfrak{R}$ are $\mathscr{V}$-categories (= $R$-linear categories or "$R$-algebras with several objects").
Hom categories are the $\mathscr{V}$-functor $\mathscr{V}$-categories

$$\mathfrak{R}(\mathscr{A}, \mathscr{B}) = [\mathscr{B}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}] \ ;$$

objects of these homs are called *modules from $\mathscr{A}$ to $\mathscr{B}$*.
These homcategories are all abelian $R$-linear categories so, within them, exact sequences have meaning.
Module composition

$$\mathfrak{R}(\mathscr{B}, \mathscr{C}) \otimes \mathfrak{R}(\mathscr{A}, \mathscr{B}) \xrightarrow{\circ} \mathfrak{R}(\mathscr{A}, \mathscr{C})$$

is defined by " tensoring over $\mathscr{B}$ ":

$$(K \circ H)(C, A) = \int^{B} H(B, A) \otimes K(C, B) \ .$$

# More on $\mathfrak{R}$

- For $\mathscr{V}$-functors $\mathscr{A} \xrightarrow{F} \mathscr{C} \xleftarrow{G} \mathscr{B}$, we have the module $\mathscr{C}(G, F) : \mathscr{A} \to \mathscr{B}$ with components $\mathscr{C}(G, F)(B, A) = \mathscr{C}(GB, FA)$.
- In particular, for each $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ we have the module $F_* = \mathscr{B}(1_{\mathscr{B}}, F) : \mathscr{A} \to \mathscr{B}$.
- Also, the module $F^* = \mathscr{B}(F, 1_{\mathscr{B}}) : \mathscr{B} \to \mathscr{A}$ provides a right adjoint $F_* \dashv F^*$ for $F_*$ in $\mathfrak{R}$.
- If $\mathscr{B}$ is Cauchy complete then every module from $\mathscr{A}$ to $\mathscr{B}$ with a right adjoint in $\mathfrak{R}$ is isomorphic to $F_*$ for some $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$.

# Lemma on retracts of adjunctions

This lemma below will be applied in $\mathfrak{R}$. It says that, if we have an idempotent 2-morphism on a morphism $f$ with an adjoint $u$, then the splitting of that idempotent on $f$ is a morphism with an adjoint obtained by splitting the mate idempotent on $u$. The form of the counit and unit is important. (A string proof is quite appealing.)

## Lemma

*In any bicategory $\mathfrak{M}$, suppose $f \dashv u : A \to X$ is an adjunction with counit $\varepsilon : fu \Rightarrow 1_A$ and unit $\eta : 1_X \Rightarrow uf$. Suppose $\omega : f \Rightarrow f$ is an idempotent 2-morphism on $f$ with splitting provided by $\sigma : f \Rightarrow g$ and $\rho : g \Rightarrow f$: that is, $\rho\sigma = \omega$ and $\sigma\rho = 1_g$. The mate $\tilde{\omega} : u \Rightarrow u$ of $\omega : g \Rightarrow g$ under $f \dashv u$ is an idempotent 2-morphism on $g$ satisfying $\varepsilon(\omega u) = \varepsilon(f\tilde{\omega})$. Then, any splitting of $\tilde{\omega}$ delivers a right adjoint $v$ for $g$. Explicitly, if $\tilde{\rho}\tilde{\sigma} = \tilde{\omega}$ and $\tilde{\sigma}\tilde{\rho} = 1_v$ then $\tilde{\varepsilon} = \varepsilon(\rho\tilde{\rho})$ and $\tilde{\eta} = (\tilde{\sigma}\sigma)\eta$ provide a counit and unit for $g \dashv v$.*

# Modules as kernels of functorial transformations

- Every module $H : \mathscr{A} \to \mathscr{B}$ is the "kernel" of a functor $\widehat{H} : [\mathscr{B}, \mathscr{V}] \to [\mathscr{A}, \mathscr{V}]$ defined by composition with $H$ in $\mathfrak{R}$ thus: $\widehat{H}(T) = (\mathscr{A} \xrightarrow{H} \mathscr{B} \xrightarrow{T} R\mathbf{1})$. That is,

$$\widehat{H}(T)A = \int^{B} H(B, A) \otimes TB \ . \qquad (1)$$

- We obtain a pseudofunctor $\widehat{(-)} : \mathfrak{R}^{\mathrm{op}} \to \mathscr{V}\text{-}\mathrm{Cat}$.
- Moreover, $\widehat{H}$ has a right adjoint $\widetilde{H} : [\mathscr{A}, \mathscr{V}] \to [\mathscr{B}, \mathscr{V}]$ given by right extension along $H$ in $\mathfrak{R}$. Explicitly,

$$\widetilde{H}(F)B = \int_{A} [H(B, A), FA] = [\mathscr{A}, \mathscr{V}](H(B, -), F) \ . \qquad (2)$$

The inclusion $j : R\mathscr{G} \hookrightarrow R\mathscr{F}$ gives the right adjoint $\mathscr{V}$-module $j^* : R\mathscr{F} \to R\mathscr{G}$ where $j^*(A, B) = R\mathscr{F}(jA, B)$.

## Lemma

*For $g \in \mathscr{G}(C, A)$ and $h \in \mathscr{F}(B, D)$, there is a commutative square*

$$\begin{array}{ccc} R\mathscr{M}'(A,B) & \xrightarrow{\subseteq} & R\mathscr{F}(A,B) \\ \downarrow & & \downarrow R\mathscr{F}(g,h) \\ R\mathscr{M}'(C,D) & \xrightarrow[\subseteq]{} & R\mathscr{F}(C,D) \end{array} \tag{3}$$

*so that $M'(A, B) = R\mathscr{M}'(A, B)$ defines a submodule $M'$ of $j^*$. There is a short exact sequence*

$$0 \longrightarrow M' \xrightarrow{\subseteq} j^* \xrightarrow{q} M \longrightarrow 0 \tag{4}$$

*in $\mathfrak{R}(R\mathscr{F}, R\mathscr{G})$ where $M(A, B) = R\mathscr{M}(jA, B)$.*

Dually, there is a short exact sequence

$$0 \longrightarrow E' \xrightarrow{\quad\subseteq\quad} j_* \xrightarrow{\quad\tilde{q}\quad} E \longrightarrow 0 \tag{5}$$

in $\mathfrak{R}(R\mathscr{G}, R\mathscr{F})$ where $E(B, A) = R\mathscr{E}(B, jA)$.

So now we have modules in both directions

$$M : R\mathscr{F} \to R\mathscr{G} \;\text{ and }\; E : R\mathscr{G} \to R\mathscr{F}\;.$$

Do they give an equivalence in $\mathfrak{R}$?
Not in general, however:

# $M \circ E \xrightarrow{\cong} 1_{R\mathscr{G}}$

## Lemma

*The family of R-module morphisms*

$$\phi_B : E(B, D) \otimes M(C, B) \to R\mathscr{G}(C, D) \ ,$$

*defined by*

$$\phi_B(e \otimes m) = \begin{cases} em & \text{for } em \in \mathscr{G} \\ 0 & \text{otherwise} \end{cases}$$

*for* $e \in \mathscr{E}(B, D)$ *and* $m \in \mathscr{M}(C, B)$*, is dinatural in* $B \in \mathscr{F}$*, natural in* $C, D \in \mathscr{G}$*, and induces an invertible morphism*

$$\bar{\phi} : M \circ E \xrightarrow{\cong} 1_{R\mathscr{G}}$$

*in* $\mathfrak{R}$.

### Corollary

*The isomorphism $\bar{\phi}$ induces an isomorphism $\widehat{E} \circ \widehat{M} \cong 1_{[\mathscr{G},\mathscr{V}]}$. The mate of this isomorphism, under the adjunction $\widehat{E} \dashv \widetilde{E}$, yields a natural transformation*

$$\Theta : \widehat{M} \to \widetilde{E} \ .$$

### Corollary

*Suppose $\Theta$ too is invertible. Then*

(i) $\widehat{E} \dashv \widehat{M}$;

(ii) $\widehat{M}$ *is fully faithful;*

(iii) $\bar{\phi} : M \circ E \xrightarrow{\cong} 1_{R\mathscr{G}}$ *is the counit of an adjunction* $M \dashv E$ *in* $\mathfrak{R}$;

(iv) $\widetilde{E} \dashv \widetilde{M}$.

There are alternative formulas for $\widehat{M}$ and $\widetilde{E}$ which can be useful. For $A \in \mathscr{F}$, let $\mathrm{Sub}A$ and $\mathrm{Quo}A$ denote representative sets of the isomorphism classes of $\mathscr{M}/A$ and $A/\mathscr{E}$, respectively. Then

$$\widehat{M}(T)A \;\cong\; \sum_{(U \xrightarrow{m} A) \in \mathrm{Sub}A} TU \quad \text{and} \quad \widetilde{E}(T)A \;\cong\; \prod_{(A \xrightarrow{e} W) \in \mathrm{Quo}A} TW \;. \tag{6}$$

In cases where $\mathrm{Sub}A$ and $\mathrm{Quo}A$ are finite, both the sum and product in these alternative formulas are direct sums and the components of $\Theta : \widehat{M} \to \widetilde{E}$ transport across the isomorphisms to matrices whose non-zero entries have the form $T(em) : TU \to TW$ for $em \in \mathscr{G}$. Suppose that elements of each $\mathrm{Sub}A$ can be listed $m_0, \ldots, m_s$ and those of each $\mathrm{Quo}A$ can be listed $e_0, \ldots, e_s$, in such a way that $e_i m_j \in \mathscr{G}$ implies $1 \leq i \leq j \leq s$ and that $e_i m_i = 1$ for $0 \leq i \leq s$. Then these matrices are square and triangular with identities down the main diagonal, and so (because we have additive inverses in the hom $R$-modules) are invertible.

## Example

Let $\mathscr{A}$ be any category in which every morphism is a monomorphism, pullbacks exist, and each slice category $\mathscr{A}/A$ has finitely many isomorphism classes. Take $\mathscr{F} = \mathscr{A}\sharp$ to be the category of spans in $\mathscr{A}$; that is, the objects are those of $\mathscr{A}$ and the morphisms $f : A \to B$ are isomorphism classes $[f_1, U, f_2]$ of spans $A \xleftarrow{f_1} U \xrightarrow{f_2} B$ in $\mathscr{A}$, where composition uses pullback. Take $\mathscr{M}$ to consist of those spans $f$ with $f_1$ invertible and $\mathscr{E}$ to consist of those spans $f$ with $f_2$ invertible. Our machinery applies. The core groupoid $\mathscr{G}$ of $\mathscr{F}$ is isomorphic to the core groupoid of $\mathscr{A}$. An example of this is $\mathscr{A} = \mathrm{FI}$, the category of finite sets and injective functions; so $\mathscr{F} = \mathrm{FI}\sharp$. Then $\mathscr{G} = \mathfrak{S}$, the groupoid of finite sets and bijections, called *the symmetric groupoid*: so $[\mathscr{G}, \mathscr{V}]$ is the category of $R$-linear Joyal species; it is equivalent to the product over $n \geq 0$ of the categories of $R$-linear representations of the groups $\mathfrak{S}_n$. The deduced equivalence $[\mathrm{FI}\sharp, \mathscr{V}] \simeq [\mathfrak{S}, \mathscr{V}]$ is in Teimuraz Pirashvili (2000).

# Plan to apply lemma on retracts of adjunctions

We continue with a proper factorization system $(\mathscr{E}, \mathscr{M})$ on a category $\mathscr{F}$. The goal is to find other conditions under which $M : R\mathscr{F} \to R\mathscr{G}$ in $\mathfrak{R}$ is an equivalence. If this is to be the case, $M$ will need to have a left adjoint so we would like to apply our Lemma to deduce the adjoint from the adjunction $j_* \dashv j^*$. This means we would like $M$ to be a retract of $j^*$. A natural splitting of the short exact sequence (4) would suffice.
A splitting $\rho : M \to j^*$ of the epimorphism $q : j^* \to M$ has components

$$\rho_{A,B} : M(A, B) \to R\mathscr{F}(A, B)$$

which are natural in $A \in \mathscr{G}$ and $B \in \mathscr{F}$. In particular they are natural in $B \in R\mathscr{M}$ and so, by Yoneda, are determined by the value, say $p_A \in R\mathscr{F}(A, A)$, at $1_A \in R\mathscr{M}(A, A)$. Then $\rho_{A,B}(m) = mp_A$. This motivates the following assumption.

**Idempotent Axiom.** For each object $A \in \mathscr{F}$, there is a morphism $p_A : A \to A$ in $R\mathscr{F}$ such that

p0. $p_A p_A = p_A$;

p1. if $f \in \mathscr{F}(A, B)$ and $f \in \mathscr{M}'$ then $f p_A = 0$;

p2. if $f \in \mathscr{F}(A, B)$ and $f \in \mathscr{E}'$ then $p_B f = 0$;

p3. if $g \in \mathscr{G}(A, B)$ then $g p_A = p_B g$;

p4. $p'_A = 1_A - p_A \in R(\mathscr{E}'(A, A) \cap \mathscr{M}'(A, A))$.

### Example

Every groupoid $\mathscr{G}$ provides a trivial example with $\mathscr{F} = \mathscr{E} = \mathscr{M} = \mathscr{G}$ and $p_A = 1_A$. In the general case, we can think of $p'_A = 1_A - p_A$ as an obstruction to $\mathscr{F}$'s being a groupoid although we will see below the sense in which $\mathscr{F}$ is not too far from $\mathscr{G}$ at the $R$-linear level.

# The retraction $\sigma$

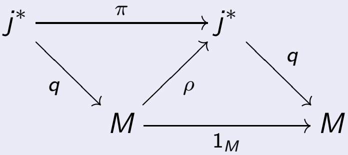
Using p1, we have the identity linear function as the left side of a commutative square (7). Since the factorization system is proper, if $u \in \mathscr{M}'(A, A)$ and $f \in \mathscr{F}(A, B)$ then $fu \in \mathscr{M}'$; so, by p4, $fp'_A \in R\mathscr{M}'(A, B)$ for all $f \in \mathscr{F}(A, B)$. This gives a splitting $\sigma_{A,B}$ of the idempotent $R\mathscr{F}(p'_A, 1_B)$.

$$\begin{array}{ccc} R\mathscr{M}'(A,B) & \xrightarrow{\subseteq} & R\mathscr{F}(A,B) \\ {\scriptstyle 1}\downarrow & \swarrow{\scriptstyle \sigma_{A,B}} & \downarrow{\scriptstyle R\mathscr{F}(p'_A, 1_B)} \\ R\mathscr{M}'(A,B) & \xrightarrow[\subseteq]{} & R\mathscr{F}(A,B) \end{array} \tag{7}$$

Using p3, we see that the linear functions $R\mathscr{F}(p'_A, 1_B)$ are the components of an idempotent module morphism $\pi' : j^* \to j^*$ which is split by the module morphism $\sigma : j^* \to M'$ and the inclusion $M' \hookrightarrow j^*$. It follows that the short exact sequence for $M$ splits so that, if we put $\pi = 1 - \pi' = R\mathscr{F}(p_A, 1_B)$, we have:

## Lemma

*The idempotent module morphism* $\pi = R\mathscr{F}(p_A, 1_B) : j^* \to j^*$ *splits as*




(8)

*where, for* $m \in \mathscr{M}(A, B)$, $\rho_{A,B}(m) = mp_A$ *and for* $g \in \mathscr{G}(A', A)$ *and* $f \in \mathscr{F}(B, B')$,

$$M(g, f)m = fmgp_{A'} \ .$$

Dually, we also have:

## Lemma

*The idempotent module morphism* $\varpi = R\mathscr{F}(1_A, p_B) : j_* \to j_*$ *splits as*

$$\begin{array}{ccccc} j_* & & \xrightarrow{\quad\varpi\quad} & & j_* & & \\ & \searrow^{\tilde{q}} & & \nearrow_{\tilde{\rho}} & & \searrow^{\tilde{q}} & \\ & & E & & \xrightarrow[\quad 1_E \quad]{} & & E \end{array} \tag{9}$$

*where* $E(A, B) = R\mathscr{E}(A, B)$, $\tilde{\rho}_{A,B}(e) = p_B e$, *and*

$$E(f, g)e = p_{B'} gef \ ,$$

*for* $e \in \mathscr{E}(A, B)$, $f \in \mathscr{F}(A', A)$, $g \in \mathscr{E}((B, B')$.

# Mates splitting

## Lemma

*The idempotent module morphisms $\pi : j^* \to j^*$ and $\varpi : j_* \to j_*$ are mates under the adjunction $j_* \dashv j^*$ in $\mathfrak{R}$. By the retract-of-adjunction Lemma, $E \dashv M$ in $\mathfrak{R}$.*

Since $\widehat{(-)} : \mathfrak{R}^{\mathrm{op}} \to \mathscr{V}\text{-}\mathrm{Cat}$ and $\widetilde{(-)} : \mathfrak{R}^{\mathrm{co}} \to \mathscr{V}\text{-}\mathrm{Cat}$ are pseudofunctors:

## Corollary

*The functor $\widehat{M} : [\mathscr{G}, \mathscr{V}] \to [\mathscr{F}, \mathscr{V}]$ has right adjoint $\widehat{E}$. The functor $\widetilde{M} : [\mathscr{F}, \mathscr{V}] \to [\mathscr{G}, \mathscr{V}]$ has right adjoint $\widetilde{E}$.*

## Corollary

*From $\widehat{M} \dashv \widetilde{M}$ and $\widehat{E} \dashv \widetilde{E}$, it follows that $\widehat{E} \cong \widetilde{M}$.*

## Lemma

*The unit* $1_{R\mathscr{G}} \Rightarrow M \circ E$ *of the adjunction* $E \dashv M$ *in* $\mathfrak{R}$ *is invertible. So the unit* $1_{[\mathscr{G},\mathscr{V}]} \Rightarrow \widehat{E} \circ \widehat{M}$ *of* $\widehat{M} \dashv \widehat{E}$ *is invertible and* $\widehat{M}$ *is fully faithful.*

## Proof.

Note that $(j^* \circ j_*)(C, D) \cong R\mathscr{F}(jC, jD)$ and, from the adjunction retract lemma, the unit of the adjunction $E \dashv M$ has component at $(C, D)$ equal to the composite

$$R\mathscr{G}(C, D) \xrightarrow{j} R\mathscr{F}(jC, jD) \cong \int^{B \in \mathscr{F}} R\mathscr{F}(B, jD) \otimes R\mathscr{F}(jC, B)$$
$$\xrightarrow{\int^{B\in\mathscr{F}} q \otimes \tilde{q}} (M \circ E)(C, D)$$

which is inverse to the component of our earlier isomorphism $\bar{\phi}$. □

# The hardest bit

## Lemma

*If each object $A$ of $\mathscr{F}$ has only finitely many $\mathscr{M}$-subobjects (that is, each slice category $\mathscr{M}/A$ has only finitely many isomorphism classes) then the counit of the adjunction $E \dashv M$ in $\mathfrak{R}$ is a split epimorphism.*

## Proof.

The counit, which is natural, is the composite

$$
\begin{aligned}
(E \circ M)(A, B) = \int^{C \in \mathscr{G}} & R\mathscr{M}(C, B) \otimes R\mathscr{E}(A, C) \\
& \to (j_* \circ j^*)(A, B) \to R\mathscr{F}(A, B)
\end{aligned}
$$

which takes the equivalence class of $m \otimes e \in R\mathscr{M}(C, B) \otimes R\mathscr{E}(A, C)$ to $mp_C e \in R\mathscr{F}(A, B)$. We claim this family has a natural right inverse.
*To be continued.* □

## Proof continued.

In particular, for $A = B$, we must see that the identity $1_A$ of $A$ is in the image. Take a finite family $(C_i \xrightarrow{m_i} A)_{i=0}^k$ of morphisms in $\mathscr{M}$ representing all isomorphism classes in the ordered set $\mathscr{M}/A$ and with the property that $m_i = m_j n$ for some $C_i \xrightarrow{n} C_j$ implies $i \leq j$; we can suppose $C_k = A$ and $m_k = 1_A$. Let

$$\phi_j : \bigoplus_{i \leq j} R\mathscr{M}(C_i, A) \otimes R\mathscr{E}(A, C_i) \longrightarrow R\mathscr{F}(A, A)$$

denote the function defined by $\phi_j(m \otimes e) = mp_{C_i}e$ for $m \in R\mathscr{M}(C_i, A)$ and $e \in R\mathscr{E}(A, C_i)$. So the image of $\phi_i$ is contained in the image of $\phi_j$ for $i \leq j$. By induction on $j$ one can show that every $f \in \mathscr{F}(A, A)$ is in the image of the function $\phi_k$.

Hence, $1_A \in \mathscr{F}(A, A)$ is in the image of the $A, A$ component of the counit.

*To be continued.* □

### Proof continued further.

Suppose $t \in \int^{C\in\mathscr{G}} R\mathscr{M}(C,A) \otimes R\mathscr{E}(A,C)$ maps to $1_A \in \mathscr{F}(A,A)$. By the Yoneda Lemma, there exists a unique family of morphisms

$$R\mathscr{F}(A,B) \longrightarrow \int^{C\in\mathscr{G}} R\mathscr{M}(C,B) \otimes R\mathscr{E}(A,C)$$

which is natural in $B \in \mathscr{F}$ taking $1_A \in \mathscr{F}(A,A)$ to $t$. Again by Yoneda, this gives right inverses to the components of the counit. Dually, there is such a family natural in $B$. By Yoneda yet again, these families agree since they have the same value at the identity. So the counit of $E \dashv M$ has a natural right inverse.

*That's really the end of proof!* □

# Main Theorem

## Theorem

*If each object $A$ of $\mathscr{F}$ has only finitely many $\mathscr{M}$-subobjects then the adjunction $E \dashv M$ is an equivalence $R\mathscr{F} \simeq R\mathscr{G}$ in $\mathfrak{R}$.*

## Proof.

We now know that the unit $1_{R\mathscr{G}} \Rightarrow \widetilde{M} \circ \widehat{M}$ is invertible. So $\widehat{M} : [R\mathscr{G}, \mathscr{V}] \to [R\mathscr{F}, \mathscr{V}]$ is fully faithful and so is its composite $M' : R\mathscr{G}^{\mathrm{op}} \to [R\mathscr{F}, \mathscr{V}]$, $C \mapsto M(C, -)$, with the Yoneda embedding. Moreover, one of our Lemmas implies $M'$ is strongly generating. By Theorem 2 of Day-St [29] (about strong generators being dense), $M'$ is also dense. So the counit of $E \dashv M$ is invertible. □

In the terminology of ring theory, this Theorem implies that $R\mathscr{F}$ and $R\mathscr{G}$ are Morita equivalent several-object $R$-algebras. In the terminology of enriched category theory, it implies that $R\mathscr{F}$ and $R\mathscr{G}$ are Cauchy equivalent $\mathscr{V}$-categories.

# Consequences

## Corollary

*The functor* $\widehat{M} : [\mathscr{G}, \mathscr{V}] \to [\mathscr{F}, \mathscr{V}]$ *is an equivalence with inverse equivalence* $\widehat{E}$.

## Corollary

*The functor* $\widetilde{M} : [\mathscr{F}, \mathscr{V}] \to [\mathscr{G}, \mathscr{V}]$ *is an equivalence with inverse equivalence* $\widetilde{E}$.

# Further consequences

### Corollary

*The equivalence $\widetilde{M}$ is a retract of the restriction functor*

$$[j,1] : [R\mathscr{F}, \mathscr{V}] \to [R\mathscr{G}, \mathscr{V}] \ .$$

### Corollary

*For any R-linear category $\mathscr{X}$ in which idempotents split and finite direct sums exist, there is an equivalence*

$$[\mathscr{F}, \mathscr{X}] \simeq [\mathscr{G}, \mathscr{X}] \ .$$

# Stiffness

## Definition

In any category $\mathscr{F}$, a morphism $f : A \to B$ is called *stiff* when the only endomorphisms of $A$ through which $f$ factors are automorphisms. In other words, $f = (A \xrightarrow{u} A \xrightarrow{v} B)$ implies $u$ invertible. A morphism is *costiff* when it is stiff in the opposite category. A category is called *stiff* when the costiff and stiff morphisms are the $\mathscr{E}$ and $\mathscr{M}$ of a proper factorization system.

The category of finite sets and the category of finite-dimensional vector spaces over a field are stiff categories: costiff = surjective, stiff = injective. Any stiff endomorphism is an automorphism so a consequence of stiffness is the pigeon-hole principle:

## Proposition (PhP)

*For all objects $A$ in a stiff category $\mathscr{F}$, the inclusions $\mathscr{M}(A, A) \subseteq \mathscr{G}(A, A)$ and $\mathscr{E}(A, A) \subseteq \mathscr{G}(A, A)$ are equalities. Equivalently, if $\mathscr{G}(A, B) \neq \emptyset$ then $\mathscr{G}(A, B) = \mathscr{E}(A, B) = \mathscr{M}(A, B)$.*

# Some work of Laci Kováks

Recall that the set $\mathscr{G}'$ of morphisms is the complement of $\mathscr{G}$ in $\mathscr{F}$.
For each $C \in \mathscr{F}$, PhP implies that $R\mathscr{G}'(C,C)$ is a two-sided ideal in $R\mathscr{F}(C,C)$. In particular $R\mathscr{G}'(C,C)$ is an $R$-algebra, possibly without an identity element.
In the case where $\mathscr{F} = \mathrm{vect}_{\mathbb{F}}$ is the category of finite vector spaces over a finite field $\mathbb{F}$ and the factorization is surjective-injective $\mathbb{F}$-linear functions, Laci Kovács (1992) produced an identity element making $R\mathscr{G}'(C,C)$ a unital algebra provided the characteristic of $\mathbb{F}$ is invertible in $R$; also see Subsection 3.1 of Kuhn's paper for helpful material on this.

Therefore we make the following definition in the general situation.

**Definition**

A *Kováks idempotent* is an identity element $\ell_C$ making $R\mathscr{G}'(C,C)$ an $R$-algebra under composition.

# From Kováks to the Idempotent Axiom

### Lemma

*Assume PhP and that each object $C$ has a Kováks idempotent $\ell_C$. Then $\ell_C$ is a central idempotent in the $R$-algebra $R\mathscr{F}(C,C)$ and the morphisms $\ell_C : C \to C$ are the components of a natural endomorphism of the identity functor of $R\mathscr{G}$.*

### Proposition

*Assume that $\mathscr{F}$ is stiff and that each object $C$ has a Kováks idempotent $\ell_C$. Then the idempotents $p_A = 1_A - \ell_A : A \to A$ satisfy the Idempotent Axiom.*

# Semisimplicity

It can happen that $R\mathscr{F}(C)$ is semisimple. This is true when $Y = \mathscr{F}(C)$ is any finite monoid of Lie type and $R$ is a field of characteristic 0 [Okniński-Putcha (1991)]. Steinberg's 2016 book provides iff conditions on a finite monoid $Y$ and field $R$ in order for $RY$ to be semisimple; in particular, the characteristic of $R$ should not divide the order of the group of invertible elements in the monoid $pYp$ for any idempotent $p \in Y$. Itamar Stein has proved all the endomorphism monoids of $R\Delta_{\perp}$ are semisimple for $R$ any field.

With semisimplicity of $RY$, the inclusion of every two-sided ideal $J$ of $RY$ into $RY$ splits as a left module morphism and splits as a right module morphism. The value of the left module splitting at the identity of $RY$ gives a right identity for $J$ and the value of the right module splitting at the identity of $RY$ gives a left identity for $J$. So $J$ becomes an $R$-algebra with identity. This is a source of Kovács idempotents for categories $\mathscr{F}$ with endomorphism monoids $\mathscr{F}(C)$ finite and $R\mathscr{F}(C)$ semisimple.

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# Thank you for your attention ☺