

Finding Structure in Science and Mathematics

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One can view the laws of nature as having goals and intentions to produce the complex structures that we see. But there is another, deeper, way of seeing our world. The universe is full of many chaotic phenomena devoid of any goals and intents. The structure that we see comes from the amazing ability that scientists have to act like a sieve and isolate those phenomena that have certain regularities. By examining such phenomena, scientists formulate laws of nature. There is an analogous situation in mathematics in which researchers choose a subset of structures that satisfy certain axioms. In this paper, we examine the way these two processes work in tandem and show how science and mathematics progress in this way. The paper ends with a speculative note on what might be the logical conclusion of these ideas.

The laws of nature that we find

Scientists look around the universe and see amazing structure. There are objects and processes of fantastic complexity. Every action in our universe follows exact laws of nature which are perfectly expressed in a mathematical language. These laws of nature are fine-tuned to bring about life, and in particular, intelligent life. It seems that the final goal of all these laws of nature is to create a creature that is in awe of the universe that created him. What exactly are these laws of nature and how do we find them?

The universe is so structured and orderly that we compare it to the most complicated and exact contraptions of the age. In the 18th and 19th century, the universe was compared to a perfectly working clock or watch. Philosophers then discussed the Watchmaker. In the 20th and 21st century, the most complicated object is a computer. The universe is compared to a perfectly working supercomputer. Researchers ask who programmed this computer. The analogy is taken even further with scientists wondering if we are like characters in *The Matrix* and actually a simulation.

How does one explain all this structure? What are the goals of these laws? Why do the laws seem so perfect for producing life and why are they expressed in an exact mathematical language?

One answer to these questions is Platonism (or its cousin Realism.) This is the belief that the laws of nature are objective and have always existed. They possess an exact ideal form that exists in Plato's realm. These laws are in perfect condition and they have formed the universe that we see around us. Not only do the laws of nature exist in this realm but it lives alongside all perfectly formed mathematics. This is supposed to help explain why the laws are written in the language of mathematics. Platonism leaves a lot to be desired. The main problem is that Platonism is metaphysics, not science. However, even if we were to accept it as true, many questions remain. How was this Platonic attic set up? Why does our physical universe follow these ethereal rules? How do scientists and mathematicians get access to Plato's little treasure chest of exact ideals?

The multiverse is another answer that has recently become quite fashionable. This is the belief that our universe is just one of many universes called the multiverse. Each universe has its own set of rules and its own possible structures that come along with those rules. Physicists who push the multiverse theory, believe that the laws in each universe is somewhat arbitrary. The reason why we see structure in our universe is that we happen to live in one of very few universes that have laws that can produce intelligent life. While the multiverse explains some of the structure that we see, there are questions that are left open. Rather than asking why the universe has any structure at all, we can push the question back and ask why the multiverse has any structure at all. Another problem is that while the multiverse would answer some of the questions we posed if it existed, who says it actually exists? Since we have no contact with possible other universes, the question of the existence of the multiverse is essentially metaphysics.

There is another, more interesting, explanation for the structure that is the focus of this paper. Rather than saying that the universe is very structured, say that the universe is chaotic and lacks structure. The reason why we see so much structure is that scientists act like a sieve and pull out only those phenomena that are predictable. They do not take into account all phenomena; rather, they select those phenomena they can deal with.

Some people say that science studies physical phenomena. This is simply not true. The exact shape of a cloud is a physical question that no scientist would try to describe. Who will win the next election is a physical question but no hard scientists would venture to give an absolute prediction. Whether or not a computer will halt for a given input can be seen as a physical question and yet we learned from Alan Turing that this question cannot be answered. Science does not study all physical phenomena. Rather, science studies predictable physical phenomena. It is almost a tautology: science predicts predictable phenomena.

Scientists have described the criteria for which phenomena they select: it is called *symmetry*. Symmetry is the property that despite something changing, there is some part of it that still remains

the same. When you say that a face has symmetry, you mean that if the left side is swapped with the right side, it will still look the same. When physicists use the word symmetry they are discussing collections of physical phenomena. A set of phenomena has symmetry if it is the same after some change. The most obvious example is symmetry of location. This means that if one performs an experiment in two different places, the results should be the same. Symmetry of time means that the outcomes of experiments should not depend on when the experiment took place. There are many other types of symmetry.

If phenomena are to be selected by scientists, then they must have many different types of symmetry. When a physicist sees a lot of phenomena, she must first determine if these phenomena have symmetry. She performs experiments in different places and at different times. If she achieves the same results, she then studies them to find the underlying cause. In contrast, if it failed to be symmetric, it would be ignored by the scientist.

The power of symmetry was first truly exploited by Albert Einstein. He postulated that the laws of physics should be the same even if the experimenter is moving close to the speed of light. With this symmetry requirement in mind, he was able to compose the laws of special relativity. Einstein was the first to understand that symmetry was the defining characteristic of physics. Whatever has symmetry will have a law of nature. The rest is not part of science.

A little after Einstein showed the importance of symmetry for the scientific endeavor, Emmy Noether proved an important theorem that established a connection between symmetry and conservation laws. Again, if there is symmetry, then there will be conservation laws. The physicists must be a sieve and allow the phenomena that do not possess symmetry to slip through her fingers.

There is actually something deeper going on here. The laws of physics cannot be found without “bracketing out” different phenomena. Consider the way the laws of physics are given as they are taught in any physics class. While they are not exactly false, they are totally useless! Ponder the simple law that Newton taught us about gravity. The force between two objects is given by the product of the two masses divided by the square of the distance. That is,

$$F = G \frac{m_1 m_2}{d^2}.$$

This is only useful when the two bodies are spherically symmetric. The bodies also have to be totally homogenous (the mass must be evenly distributed). There cannot be any third body or gravitational forces affecting either of the bodies. Both bodies must be magnetically and electrically neutral. They cannot be traveling near the speed of light (lest the theory of special relativity take over). The bodies

also cannot be too small (lest quantum effects come into play.) And the list goes on. In summation, it is safe to say that there were probably never two bodies that exactly satisfied the requirements for Newton's laws to be exactly true. Rather, the physicists must make controlled experiments and ignore all these other effects. By selecting the phenomena, he idealizes the actual world to find the ideal laws of nature. Without selecting the phenomena, no such laws can be found.

No one is asserting that selecting subsets of phenomena is the only way of finding laws of nature. There are other methods for finding such laws. For example, in statistical mechanics and in quantum theory, one considers large ensembles of phenomena to be one phenomenon (a quotient set of phenomena, rather than a subset). While we acknowledge the existence of other methods, in this paper we will focus on our selection method.

There are a few problems with this explanation of the structure found in the universe. For one, it seems that phenomena that we do select and that have laws of nature are exactly the phenomena that generate *all* the phenomena. So, while the shape of a cloud or the winner of an election are too complicated for the scientist to worry about, they are generated by laws of water molecules and brain synapses that are part of science. Where is the boundary between science and non-science?

Despite these failings of our explanation for the structure, we believe it is the best candidate for being *the* solution. It is one of the only solutions that does not invoke any metaphysical principle or the existence of a multitude of unseen universes. We do not have to look outside the universe to find a cause for the structure that we find in the universe. Rather, we look at how we are looking at phenomena.

Before we move on, we should point out that our solution has a property in common with the multiverse solution. We postulated that, for the most part, the universe is chaotic and there is not so much structure in it. We, however, focus only on the small amount of structure that there is. Similarly, one who believes in the multiverse believes that most of the multiverse lacks structure. It is only in a select few universes that we do find any structure. And we inhabitants of this structured universe are focused on that rare structure. Both solutions are about focusing on the small amount of structure in a chaotic whole.

A hierarchy of number systems

This idea that we only see structure because we are focusing on a subset of phenomena is novel and hard to wrap one's head around. There is an analogous situation in mathematics that is much easier to understand. We will focus on one important example where on this selection process is seen very

clearly. First we need to take a little tour of several number systems and their properties.

Consider the real numbers. In the beginning of high school, the teacher draws the real number line on the board and says that these are all the numbers one will ever need. Given two real numbers, we know how to add, subtract, multiply and divide them. They comprise a number system that is used in every aspect of science. The real numbers also have an important property: they are totally ordered. That means that given any two different real numbers, one is less than the other. Just think of the real number line: given any two different points on the line, one will be to the right of the other. This property is so obvious that it is barely mentioned.

While the real numbers seem like a complete picture, the story does not end there. Already in the 16th century, mathematicians started looking at more complicated number systems. They began working with an “imaginary” number i which has the property that when it is squared it is -1 . This is in stark contrast to any real number whose square is never negative. They defined an imaginary number as the product of a real number and i . Mathematicians went on to define a complex number that is the sum of a real number and an imaginary number. If r_1 and r_2 are real numbers, then $r_1 + r_2i$ is a complex number. Since a complex number is built from two real numbers and we usually draw them in a two-dimensional plane. The real number line sits in the complex plane. This corresponds to the fact that every real number, r_1 , can be seen as the complex number $r_1 + 0i$.

We know how to add, subtract, multiply, and divide complex numbers. However, there is one property that is different about the complex numbers. In contrast to the real numbers, the complex numbers are not totally ordered. Given two complex numbers, say $3 + 7.2i$ and $6 - 4i$, can we tell which one is more and which one is less? There is no obvious answer. (In fact, one can totally order the complex numbers but the ordering will not respect the multiplication of complex numbers.) The fact that the complex numbers are not totally ordered means that we lose structure when we go from the real numbers to the complex numbers.

The story is not over with the complex numbers. Just as one can construct the complex numbers from pairs of real numbers, so too, one can construct the quaternions from pairs of complex numbers. Let $c_1 = r_1 + r_2i$ and $c_2 = r_3 + r_4i$ be complex numbers; then we can construct a quaternion as $q = c_1 + c_2j$ where j is a special number. It turns out that every quaternion can be written as

$$r_1 + r_2i + r_3j + r_4k,$$

where i , j , and k are special numbers. So while the complex numbers are comprised of two real numbers, the quaternions are comprised of four real numbers. Every complex number $r_1 + r_2i$

can be seen as a special type of quaternion: $r_1 + r_2\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. We can think of the quaternions as a four-dimensional space which has the complex numbers as a two-dimensional subset of it. We humans have a hard time visualizing such higher-dimensional spaces.

The quaternions are a full-fledged number system. They can be added, subtracted, multiplied and divided with ease. Like the complex numbers, they fail to be totally ordered. But they have even less structure than the complex numbers. While the multiplication of complex numbers is commutative, that is, for all complex numbers c_1 and c_2 we have that $c_1c_2 = c_2c_1$, this is not true for all quaternions. This means there are quaternions q_1 and q_2 such that q_1q_2 is different than q_2q_1 .

This process of doubling a number system is called the “Cayley–Dickson construction,” named after the mathematicians Arthur Cayley and Leonard Eugene Dickson. Given a certain type of number system, one gets another number system that is twice the dimension of the original system. The new system that one develops has less structure (i.e. fewer axioms) than the starting system.

If we apply the Cayley–Dickson construction to the quaternions, we get the number system called the *octonions*. This is an eight-dimensional number system. That means that each of the octonions can be written with eight real numbers as

$$r_1 + r_2\mathbf{i} + r_3\mathbf{j} + r_4\mathbf{k} + r_5\mathbf{l} + r_6\mathbf{m} + r_7\mathbf{n} + r_8\mathbf{p}.$$

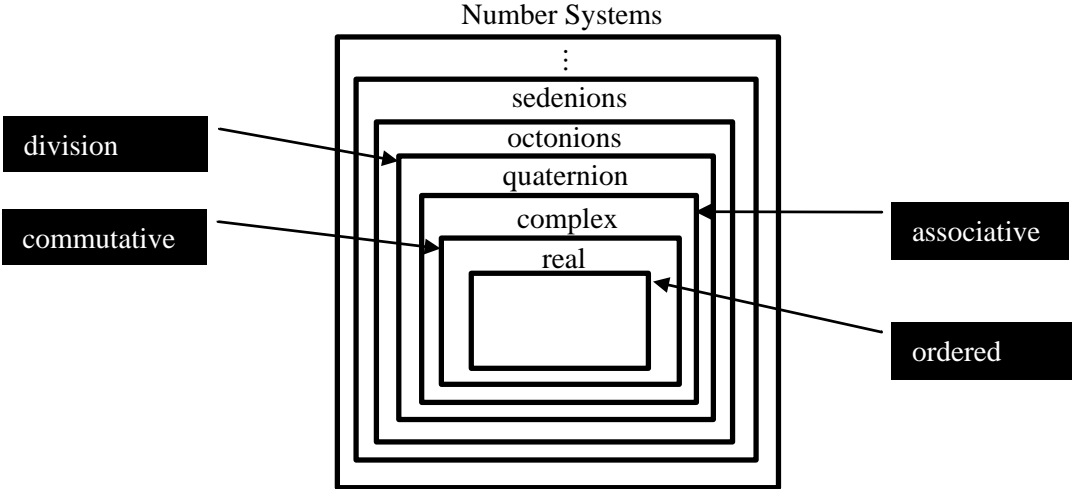
Although it is slightly complicated, we know how to add, subtract, multiply, and divide octonions. Every quaternion can be written as a special type of octonion in which the last four coefficients are zero.

Like the quaternions, the octonions are neither totally ordered nor commutative. However, the octonions also fail to be associative. In detail, all the number systems that we have so far discussed possess the associative property. This means that for any three elements, a , b , and c , the two ways of multiplying them, $a(bc)$ and $(ab)c$, are equal. However, associativity does not always work with the octonions. That is, there exists octonions o_1 , o_2 and o_3 such that $o_1(o_2o_3) \neq (o_1o_2)o_3$.

We can go on with this doubling and get an even larger, sixteen-dimensional number system called the *sedenions*. In order to describe a sedonian, one would have to give sixteen real numbers. Octonions are a special type of sedonian: their last eight coefficients are all zero. But researchers steer clear of sedenions because they lose an important property. While one can add, subtract, and multiply sedenions, there is no way to divide them nicely. Most physicists think this is beyond the pale and “just” mathematics. Even mathematicians find sedenions hard to deal with. One can go on to formulate 32-dimensional number systems and 64-dimensional number systems, etc. But they

are usually not discussed because, as of now, they do not have many applications. We will concentrate on the octonions.

Here is a diagram of all these different number systems and the properties they have:



Let us discuss the applicability of these number systems. The real numbers are used in every aspect of physics. All quantities, measurements, durations, and lengths of physical objects are given as real numbers. Although complex numbers were formulated by mathematicians to help solve equations (i is the solution to the equation $x^2 = -1$), in the middle of the 19th century, physicists started using complex numbers to discuss waves. In the 20th century, complex numbers became fundamental for the study of quantum mechanics. By now, the role of complex numbers is very important in many different branches of physics. The quaternions show up in physics but are not a major player. The octonions, the sedenions, and the larger number systems rarely arise in the physics literature.

The laws of mathematics that we find

The usual view of these number systems is to think that the real numbers are fundamental while the complex, quaternions, and octonions are strange larger sets that keep mathematicians and some physicists busy. The larger number systems seem unimportant and less interesting.

Let us turn this view on its head. Rather than looking at the real numbers as central and the octonions

as strange larger number systems, think of the octonions as fundamental and all the other number systems as just special subsets of octonions. The only number system that really exists is the octonions. To paraphrase Leopold Kronecker, "God made the octonions, all else is the work of man." The octonions contain every number that we will ever need. (And, as we stated earlier, we can do the same trick with the sedenions and even the 64-dimensional number system. We shall fix our ideas with the octonions.)

Let us explore how we can derive all the properties of the number systems that we are familiar with. Although the multiplication in the octonions is not associative, if one wants an associative multiplication, one can look at a special subset of the octonions. (We are using the word "subset" but we need a special type of subset that respects the operations of the number system. Such subsets are called "subgroups," "subfields," or "sub-normed-division-algebras," etc. For the reader's benefit, we use "subset.") So if one selects the subset of all octonions of the form

$$r_1 + r_2\mathbf{i} + r_3\mathbf{j} + r_4\mathbf{k} + 0\mathbf{l} + 0\mathbf{m} + 0\mathbf{n} + 0\mathbf{p},$$

then the multiplication will be associative (like the quaternions). If one further looks at all the octonions of the form

$$r_1 + r_2\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + 0\mathbf{l} + 0\mathbf{m} + 0\mathbf{n} + 0\mathbf{p},$$

then the multiplication will be commutative (like the complex numbers). If one further selects all the octonions of the form

$$r_1 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + 0\mathbf{l} + 0\mathbf{m} + 0\mathbf{n} + 0\mathbf{p},$$

then they will have a totally ordered number system. All the axioms that one wants satisfied are found "sitting inside" the octonions.

This is not strange. Whenever we have a structure, we can focus on a subset of special elements that satisfies certain properties. Take, for example, any group. We can go through the elements of the group and pick out those X such that, for all elements Y , we have that $XY = YX$. This subset is a commutative (abelian) group. That is, it is a fact that in any group there is a subset which is a commutative group. We simply select those parts that satisfy the axiom and ignore ("bracket out") those that do not. The point we are making is that if a system has a certain structure, special subsets of that system will satisfy more axioms than the starting system.

This is similar to what we are doing in physics. We do not look at all phenomena. Rather, we pick

out those phenomena that satisfy the requirements of symmetry and predictability. In mathematics, we describe the subset with the axiom that describes it. In physics, we describe the selected subset of phenomena with a law of nature.

Working in tandem and moving forward

We have shown that there is an important analogy between physics and mathematics. In both fields, if we do not look at the entirety of a system, but rather look at special subsets of the system, we see more structure. In physics we select certain phenomena (the ones that have a type of symmetry) and ignore the rest. In mathematics we are looking at certain subsets of structures and ignore the rest. These two bracketing operations work hand in hand.

The job of physics is to describe a function from the collection of observed physical phenomena to mathematical structure:

observed physical phenomena \rightarrow mathematical structure.

That is, we have to give mathematical structure to the world we observe. As physics advances and we try to understand more and more observed physical phenomena, we need larger and larger classes of mathematics. In terms of this function, if we are to enlarge the input of the function, we need to enlarge the output of the function.

Some examples of this broadening of physics and mathematics are needed. (i) When physicists started working with quantum mechanics they realized that the totally ordered real numbers are too restrictive for their needs. They required a number system with fewer axioms. They found the complex numbers. (ii) When Albert Einstein wanted to describe general relativity, he realized that the mathematical structure of Euclidean space with its axiom of flatness (Euclid's fifth axiom) was too restrictive. He needed curved, non-Euclidian space to describe the space-time of general relativity. (iii) In quantum mechanics it is known that for some systems, if we first measure X and then Y, we will get different results than first measuring Y and then measuring X. In order to describe this situation mathematically, one needed to leave the nice world of commutativity. They required the larger class of structures where commutativity is not assumed. (iv) When Boltzmann and Gibbs started talking about statistical mechanics, they realized that the laws they were coming up with were no longer deterministic. Outcomes of experiments no longer either happen ($p(X) = 1$) or do not happen ($p(X) = 0$). Rather, with statistical mechanics one needs probability theory. The chance of a certain outcome of an experiment is a probability ($p(X)$) is an element of the infinite set $[0,1]$ rather

than the restrictive finite subset $\{0,1\}$). (v) When scientists started talking about the logic of quantum events, they realized that the usual logic, which is distributive, is too restrictive. They needed to formulate the larger class of logics in which the distributive axiom does not necessarily hold true. This is now called quantum logic. Many other examples exist.

Paul A.M. Dirac understood this loosening of axioms about 85 years ago when he wrote the following:

The steady progress of physics requires for its theoretical formulation a mathematics which get continually more advanced. This is only natural and to be expected. What however was not expected by the scientific workers of the last century was the particular form that the line of advancement of mathematics would take, namely it was expected that mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundation and gets more abstract. Non- euclidean geometry and noncommutative algebra, which were at one time were considered to be purely fictions of the mind and pastimes of logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and the advance in physics is to be associated with continual modification and generalisation of the axioms at the base of mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.

As physics progresses and we become aware of more and more physical phenomena, larger and larger classes of mathematical structures are needed and we get them by looking at fewer and fewer axioms. There is no doubt that if Dirac lived now, he would talk about the rise of octonions and even the sedenions within the needed number systems.

In order to describe more phenomena, we will need larger and larger classes of mathematical structures and hence fewer and fewer axioms. What is the logical conclusion to this trend? How far can this go? Physics wants to describe more and more phenomena in our universe. Let us say we were interested in describing *all* phenomena in our universe. That is, we don't "bracket out" any phenomena. What type of mathematics would we need? How many axioms would be needed for the mathematical structure to describe all the phenomena? Of course, it is hard to predict but it is even harder not to speculate. One possible conclusion would be that if we look at the universe in totality and not bracket any subset of phenomena, the mathematics we would need would have no

axioms at all. That is, the universe in totality is devoid of structure and needs no axioms. Total Lawlessness! This would finally eliminate all metaphysics when dealing with the laws of nature and mathematical structure. It is only the way that we look at the universe that gives us the illusion of structure.

With this view of physics, we come to even more profound questions. These are the future projects of science. If the structure that we see is only an illusion, then why do we see this illusion? Instead of looking at the laws of nature that are formulated by scientists, we have to look at scientists and the way they pick out subsets of phenomena and their concomitant laws of nature. What is it about human beings that renders us so good at being sieves? Rather than looking at the universe, we should look at the *way* we look at the universe.

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