Homotopy type theory: a new connection between logic, category theory and topology

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Category Theory Seminar, CUNY, October 26, 2018

Classical connections between logic, algebra and topology

- Boolean algebras
- Heyting algebras
- Cylindric algebras
- Categorical doctrines
- Cartesian closed categories and λ-calculus
- Autonomous categories and linear logic

Topos theoretic connections:

- Frames and locales
- Elementary toposes and intuitionistic set theory
- Grothendieck toposes and geometric logic
- Realisability toposes

Axiomatic Homotopy Theory

J.H.C. Whitehead (1950):
The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that analytic is equivalent to pure projective geometry.

Traditional axiomatic systems in homotopy theory:

- Triangulated categories [Verdier 1963];
- Homotopical algebra [Quillen 1967];
- Derivators [Grothendieck 1984]

New axiomatic systems:

- Higher toposes [Rezk, Lurie, ...];
- Homotopy type theory [Voevodsky, Awodey & Warren,....];
- Cubical type theory [Coquand & collaborators].

Voevodsky's Univalent Foundation Program

Voevodsky's univalence principle is to type theory what the induction principle is to Peano arithmetic.

A **univalent type theory** (UTT) is obtained by adding the univalence principle to Martin-Löf type theory (MLTT).

The goal of Voevodsky's Univalent Foundation Program is to

- give constructive mathematics a new foundation;
- apply type theory to homotopy theory;
- develop proof assistants based in UTT.

But many basic questions remain to be solved.

Goals of my talk

To describe the connection between type theory, category theory and topology by using the notion of **tribe**.

Theorem

(Gambino & Garner, Shulman) The syntaxic category of type theory is a tribe.

The notion of tribe is a gateway to type theory:

We may work backward

$$Tribe \Rightarrow Type\ \ Theory$$

EVERY MATHEMATICIAN IS USING TYPE THEORY WITHOUT BEING AWARE OF IT

Overview

1. What is a tribe?
2. What is type theory ?
3. What is univalence?
4. What is descent?
5. What is an elementary higher topos ?
6. Applications

Carrable maps

Recall that a map $p : X \to B$ in a category $\mathcal{C}$ is said to be **carrable** if the fiber product of p with any map $f : A \to B$ exists,

$$\begin{tikzcd} A \times_B X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The projection π_1 is called the **base change** of p along f.

The notion of clan

Definition

A **clan** is a category $\mathcal{C}$ with terminal object 1 and equipped with a class $\mathcal{F}$ of carrable maps called **fibrations** satisfying the following conditions:

- every isomorphism is a fibration;
- the composite of two fibrations is a fibration;
- the base change of a fibration along any map is a fibration;
- the unique map $X \to 1$ is a fibration for every object $X \in \mathcal{C}$.

Definition

A **homomorphism of clans** $F : \mathcal{C} \to \mathcal{C}'$ is a functor which preserves

- fibrations and base changes of fibrations;
- terminal objects.

Anodyne maps

Definition

A map $u : A \to B$ in a clan $\mathcal{C}$ is said to be is **anodyne** if it has the left lifting property with respect to every fibration $f : X \twoheadrightarrow Y$.

This means that every commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "anodyne\ \ u"'] & X \arrow[d, two heads, "f\ \ fibration"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

has a diagonal filler $d : B \to X$ $(du = a$ and $fd = b)$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, two heads, "f"] \\ B \arrow[r, "b"] \arrow[ur, dashed, "d"] & Y \end{tikzcd}$$

Definition

A clan $\mathcal{E}$ is a **tribe** if

- the base change of an anodyne map along a fibration is anodyne;
- every map $f : A \to B$ admits a factorization $f = pu$ with u anodyne and p a fibration:

$$\begin{tikzcd} & E \arrow[dr, ->>, "p"] & \\ A \arrow[ur, "u"] \arrow[rr, "f"] & & B. \end{tikzcd}$$

Definition

A **homomorphism of tribes** $F : \mathcal{E} \to \mathcal{E}'$ is a homomorphism of clans which takes anodyne maps to anodyne maps.

The tribe of Kan complexes

A map of simplicial sets $f : X \to Y$ is said to be a **Kan fibration** if every commutative square

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "h"] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r, "y"] & Y \end{tikzcd} \qquad \Rightarrow$$

has a diagonal filler $h' : \Delta[n] \to Y$.

A simplicial set X is said to be a **Kan complex** if the map $X \to 1$ is a Kan fibration.

The tribe of Kan complexes

Theorem

The category of Kan complexes **Kan** *has the structure of a tribe, where a fibration is a Kan fibration.*

Remark: a map between Kan complexes $u : A \to B$ is anodyne iff it is a strong deformation retract.

Types and elements

Let $\mathcal{E}$ be a tribe.

We shall say that an object $A \in \mathcal{E}$ is a **type** and write

$$\vdash A : Type$$

We shall say that a map $a : 1 \to A$ in $\mathcal{E}$ is an **element** of type A and write

$$\vdash a : A$$

Remark: an element $a : A$ is often called a **term** of type A.

Elementary extension of a tribe

Let $\mathcal{E}$ be a tribe.

Then for every object $A \in \mathcal{E}$ we have a new tribe $\mathcal{E}(A)$.

By construction, $\mathcal{E}(A)$ is the full subcategory of $\mathcal{E}/A$ whose objects are fibrations $p : X \twoheadrightarrow A$.

A map $f : (X, p) \to (Y, q)$ in $\mathcal{E}(A)$ is a *fibration* if the map $f : X \to Y$ is a fibration in $\mathcal{E}$.

$$\begin{tikzcd} X \arrow[rr, two heads, "f"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\ & A & \end{tikzcd}$$

A fibration is a family of objects.

If $p : E \to A$ is a fibration, then the object (E, p) of the tribe $\mathcal{E}(A)$ is called a **dependant type in context** A.

A type theorist will write

$$x : A \vdash E(x) : Type \tag{1}$$

where $E(x)$ denotes the **fiber** of $p : E \to A$ at a *variable element* $x : A$.

$$\begin{tikzcd} E(x) \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\ 1 \arrow[r, "x"] & A. \end{tikzcd}$$

The fibration $p : E \to A$ is defining a *family* $(E(x) \mid x : A)$ of objects of $\mathcal{E}$ indexed by elements of A.

Sections of a fibration

A section $s : A \to E$ of a fibration $p : E \to A$ has a value $s(x) : E(x)$ for every element $x : A$.

$$
\begin{tikzcd}
E(x) \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\
1 \arrow[u, bend left, "s(x)"] \arrow[r, "x"] & A \arrow[u, bend left, "s"]
\end{tikzcd}
$$

A type theorist will write:

$$x : A \vdash s(x) : E(x) \tag{2}$$

The section $s : A \to E$ is defining a *family of elements* $s(x) : E(x)$ indexed by the elements $x : A$.

Elementary extension

The functor

$$e : \mathcal{E} \to \mathcal{E}(A)$$

defined by putting $e(X) = (A \times X, \pi_1)$ for every object $X \in \mathcal{E}$ is a homomorphism of tribes.

In type theory, the functor $e : \mathcal{E} \to \mathcal{E}(A)$ is defined by *context extension*:

$$\frac{\vdash B : \mathit{Type}}{x : A \vdash B : \mathit{Type}} \qquad \frac{\vdash t : B}{x : A \vdash t : B}$$

A map between two types $f : A \to B$ is an element $f(x) : B$ indexed by a variable element $x : A$.

$$x : A \vdash f(x) : B \tag{3}$$

Change of parameters

If $f : A \to B$ is a map in a tribe $\mathcal{E}$, then the base change functor

$$f^\star : \mathcal{E}(B) \to \mathcal{E}(A)$$

is a homomorphism of tribes.

In type theory, the functor $f^\star$ corresponds to the operation of *substitution*: $y := f(x)$

$$\frac{y : B \vdash E(y) : \mathit{Type}}{x : A \vdash E(f(x)) : \mathit{Type}} \qquad \frac{y : B \vdash s(y) : E(y)}{x : A \vdash s(f(x)) : E(f(x))}$$

Σ-formation rule

If A is an object in a tribe $\mathcal{E}$, then the functor $e_A : \mathcal{E} \to \mathcal{E}(A)$ has a left adjoint $\Sigma_A : \mathcal{E}(A) \to \mathcal{E}$ defined by putting $\Sigma_A(E, p) = E$.

$$\frac{x : A \vdash E(x) : Type}{\vdash \sum_{x:A} E(x) : Type}$$

More generally if $f : A \to B$ is a fibration, then the base change functor $f^\star : \mathcal{E}(B) \to \mathcal{E}(A)$ has a left adjoint $\Sigma_f : \mathcal{E}(A) \to \mathcal{E}(B)$.

$$\frac{x : A \vdash E(x) : Type}{y : B \vdash \sum_{x:A(y)} E(x) : Type}$$

Internal products

A tribe $\mathcal{E}$ has **internal products** if for every fibration $f : A \to B$ the base change functor $f^\star : \mathcal{E}(B) \to \mathcal{E}(A)$ has a right adjoint

$$\Pi_f : \mathcal{E}(A) \to \mathcal{E}(B)$$

and if

- the functor Π_f takes anodyne maps to anodyne maps;
- the Beck-Chevalley condition holds:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "pullback"] & C \arrow[d, two heads, "g"] \\ B \arrow[r, "v"] & D \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathcal{E}(A) \arrow[d, "\Pi_f"'] \arrow[dr, phantom, "comm"] & \mathcal{E}(C) \arrow[l, "u^\star"'] \arrow[d, "\Pi_g"] \\ \mathcal{E}(B) & \mathcal{E}(D) \arrow[l, "v^\star"'] \end{tikzcd}$$

The tribe of Kan complexes **Kan** has internal products.

Internal products

If $\mathcal{E}$ has internal products, then the functor $e : \mathcal{E} \to \mathcal{E}(A)$ has a right adjoint

$$\Pi_A : \mathcal{E}(A) \to \mathcal{E}$$

for every object $A \in \mathcal{E}$. In type theoretic notation:

$$\frac{x : A \vdash E(x) : Type}{\vdash \prod_{x:A} E(x) : Type}$$

The object of maps $A \to B$ between two objects $A, B \in \mathcal{E}$ is defined by putting

$$[A, B] := \prod_{x:A} B$$

An element $f : [A, B]$ is a map $f : A \to B$.

Path object

Let A be an object in a tribe $\mathcal{E}$.

A **path object** for A is obtained by factoring the diagonal $\Delta : A \to A \times A$ as an anodyne map $r : A \to PA$ followed by a fibration $(s, t) : PA \to A \times A$,

$$\begin{tikzcd} & PA \arrow[dr, "{(s,t)}"] & \\ A \arrow[ur, "r"] \arrow[rr, "\Delta"] & & A \times A. \end{tikzcd}$$

A **homotopy** $h : f \rightsquigarrow g$ between two maps $f, g : A \to B$ is a map $h : A \to PB$ such that $sh = f$ and $th = g$.

The homotopy relation $f \sim g$ is a congruence on the arrows of $\mathcal{E}$.

The homotopy category

Let $\mathcal{E}$ be a tribe

The **homotopy category** of $\mathcal{E}$ is the quotient of $\mathcal{E}$ by the homotopy relation $\sim$.

$$ho(\mathcal{E}) := \mathcal{E}/\sim$$

A map $f : X \to Y$ in $\mathcal{E}$ is called a **homotopy equivalence** if it is invertible in $ho(\mathcal{E})$.

Every anodyne map is a homotopy equivalence.

An object X is **contractible** if the map $X \to 1$ is a homotopy equivalence.

Identity type

In Martin-Löf type theory, there is a type constructor which associates to every type A a dependant type

$$x : A, y : A \ \vdash Id_A(x, y) : Type$$

called the **identity type** of A.

An element $p : Id_A(x, y)$ is regarded as a **proof** that $x =_A y$.

There is a **reflexivity term**

$$x : A \ \vdash r(x) : Id_A(x, x)$$

which proves that $x =_A x$.

The identity type is a path object

Let us put

$$Id_A = \sum_{x:A} \sum_{y:A} Id_A(x,y)$$

We obtain a factorisation of the diagonal $\Delta : A \to A \times A$

$$\begin{tikzcd} & Id_A \arrow[dr, "{(s,t)}"] & \\ A \arrow[ur, "r"] \arrow[rr, "\Delta"] & & A \times A \end{tikzcd}$$

(Awodey & Warren) The factorisation $\Delta = (s,t)r$ a path object for A. A proof $p : Id_A(x,y)$ is a homotopy $p : x \rightsquigarrow y$.

Tribes are fibration categories

A *Brown fibration* category is a clan $\mathcal{E}$ equipped with a class $\mathcal{W}$ of *equivalences* such that:

- the base change of an equivalence along a fibration is an equivalence;
- every map $f : A \to B$ admits a factorization $f = pw$ with w an equivalence and p a fibration;
- $\mathcal{W}$ satisfies 6-for-2:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "gf \in \mathcal{W}"] & B \arrow[r, "g"] \arrow[rr, bend right, "hg \in \mathcal{W}"'] & C \arrow[r, "h"] & D \end{tikzcd} \quad \Rightarrow \quad f, g, h \in \mathcal{W}$$

Theorem

(Shulman, J.) Every tribe is a Brown fibration category.

HOTT in action

Let $\mathcal{E}$ be a tribe with internal products.

Definition

(Voevodsky) If A is an object of $\mathcal{E}$ we can put

$$IsCont(A) := \sum_{y:A} \prod_{x:A} Id_A(x, y)$$

An element $p : IsCont(A)$ is a **proof** that A is contractible

This may be compared with

$$IsSingleton(A) := (\exists y \in A)(\forall x \in A)\ x = y$$

(the Curry-Howard correspondance).

The object *IsEquiv*(f)

Let $f : A \to B$ be a map in a tribe $\mathcal{E}$ (with internal products).

The *homotopy fiber* of f at $y : B$ is defined by putting

$$Fib(f)(y) := \sum_{x:A} Id_B(f(x), y)$$

Theorem

(Voevodsky) A map $f : A \to B$ is a homotopy equivalence if and only if the object

$$IsEquiv(f) := \prod_{y:B} IsCont(Fib(f)(y))$$

has an element $p : IsEquiv(f)$.

The object $Eq(A, B)$

Let A and B be two objects in a tribe (with internal products).

Definition

(Voevodsky) Let us put

$$Eq(A, B) := \sum_{f:[A,B]} IsEquiv(f)$$

An element $w : Eq(A, B)$ is a homotopy equivalence $w : A \simeq B$.

$Eq_{A\times A}(E)$

For every fibration $p : E \to A$ let us put

$$Eq_{A\times A}(E) = \sum_{x:A}\sum_{y:A} Eq(E(x), E(y))$$

We then have a factorisation

$$\begin{tikzcd} & Eq_{A\times A}(E) \arrow[dr, "{(p_1,p_2)}"] & \\ A \arrow[ur, "u"] \arrow[rr, "\Delta"] & & A \times A \end{tikzcd}$$

where $u(x) : Eq(E(x), E(x))$ represents the identity map $E(x) \to E(x)$ for every $x : A$.

Connection

Definition

A (pseudo) *connection* on a fibration $p : E \to A$ is a map $\gamma : Id_A \to Eq_{A\times A}(E)$ such that the following square commutes

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "r"'] & Eq_{A\times A}(E) \arrow[d, "(p_1,p_2)"] \\ Id_A \arrow[r, "(s,t)"'] \arrow[ur, dashed, "\gamma"] & A \times A \end{tikzcd}$$

The map $\gamma(x, y) : Id_A(x, y) \to Eq(E(x), E(y))$ takes a path $p : x \rightsquigarrow y$ to an equivalence $\gamma(p) : E(x) \simeq E(y)$.

Theorem

(Voevodsky) *Every fibration $p : E \to A$ admits a connection $\gamma : Id_A \to Eq_{A\times A}(E)$ and γ is homotopy unique.*

Univalent fibration

Definition

(Voevodsky) We say that a fibration $E \to A$ is **univalent** if the connection $\gamma : A \to Eq_{A\times A}(E)$ is a homotopy equivalence.

This means that the map

$$\gamma(x, y) : Id_A(x, y) \to Eq(E(x), E(y))$$

is a homotopy equivalence for every $x, y : A$.

A fibration $E \to A$ is univalent if and only if the unit map $u : A \to Eq_{A\times A}(E)$ is a homotopy equivalence.

Small fibrations

Let $\mathcal{E} = (\mathcal{E}, \mathcal{F})$ be a tribe.

We say that sub-class $\mathcal{F}' \subseteq \mathcal{F}$ is a class of **small fibrations** if the following conditions hold:

- every isomorphism is a small fibration;
- the base change of a small fibration along any map is small;
- If $w : (X, p) \to (Y, p)$ is a homotopy equivalence in $\mathcal{E}(A)$, then the fibration $p : X \to A$ is small if and only if the fibration $q : Y \to A$ is small;
- Etc ....

In the tribe of Kan complexes **Kan**, there is a notion of κ-small fibration for every strongly inaccessible cardinal κ.

Semi-universal fibration

We say that a small fibration $q : \mathbb{E} \to \mathbb{U}$ is **semi-universal** if for every small fibration $p : E \to A$ there exists a homotopy pullback square:

$$\begin{tikzcd} E \arrow[r, "\phi'"] \arrow[d, "p"'] & \mathbb{E} \arrow[d, "q"] \\ A \arrow[r, "\phi"] & \mathbb{U} \end{tikzcd}$$

(homotopy pullback square: the induced map $E \to A \times_{\mathbb{U}} \mathbb{E}$ is a homotopy equivalence)

(Voevodsky) If the fibration $q : \mathbb{E} \to \mathbb{U}$ is univalent, then the classifying pair (ϕ, ϕ') is homotopy unique.

Martin-Löf type theory (MLTT) has semi-universal fibrations, but they are not univalent.

Universal fibration

We shall say that a small fibration $q : \mathbb{E} \to \mathbb{U}$ is **universal** if it is semi-universal and univalent. The object $(\mathbb{U}, q)$ is said to be a *universe*.

Theorem

(Voevodsky) *The tribe of Kan complexes* **Kan** *has a universal κ-small fibration $\mathbb{E}_\kappa \to \mathbb{U}_\kappa$ for every strongly inaccessible cardinal κ.*

Univalent type theory

Definition

We shall say that the **univalence principle** holds in a tribe if for every fibration $p : E \to A$ there exists a univalent fibration $p' : E' \to A'$ together with a homotopy pullback square

$$\begin{tikzcd} E \arrow[r, "\phi'"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ A \arrow[r, "\phi"] & A' \end{tikzcd}$$

Definition

We shall say that a type theory T is **univalent** the univalence principle holds in T.

For example, Cubical Type Theory (CTT) (Thierry Coquand and collaborators) is univalent.

Homotopy colimits in a tribe

The notion of homotopy colimit is defined in type theory with the notion of **inductive type**.

References: *Inductive types in homotopy type theory* [Awodey-Gambino-Sojakova].

We shall use the notion of homotopy pushout informally.

Homotopy pushouts

Recall that a commutative square of simplicial sets

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d] & E \arrow[d] \\ B \arrow[r] & F \end{tikzcd}$$

is said to be be **homotopy pushout**, or **homotopy cocartesian** if the map $B \sqcup_A E' \to F$ in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "v'"] \arrow[d] & E' \arrow[d] \arrow[dr] & \\ B \arrow[r] & B \sqcup_A E' \arrow[r] & F \end{tikzcd}$$

is a homotopy equivalence, where $v = wv' : A \to E' \to E$ is a factorisation of v as a cofibration v' followed by homotopy equivalence w.

Descent

A cube $C : [1]^3 \to \mathcal{E}$, when viewed from above, becomes a square $C' : [1]^2 \to \mathcal{E}^{[1]}$ in the arrow category of $\mathcal{E}$.

$$\begin{tikzcd} f \arrow[r, "\beta"] \arrow[d, "\alpha"'] & h \arrow[d, "\delta"] \\ g \arrow[r, "\gamma"] & k \end{tikzcd} \tag{4}$$

An edge of C' is a square in $\mathcal{E}$.

Theorem

[Rezk] *(Descent for pushouts) Suppose that the square $C' : [1]^2 \to \Delta\mathbf{Set}^{[1]}$ is homotopy cocartesian. If the squares α and β of C' are homotopy cartesian, then so are the squares δ and γ.*

Univalence $\Rightarrow$ Descent

The proof is left as an exercise to the reader. Hint: we can suppose that the maps f, g, h and k are Kan fibrations; we then use the following diagram

$$\begin{tikzcd}
f \arrow[r, "\beta"] \arrow[d, "\alpha"'] & h \arrow[d, "\delta"'] \arrow[ddr, bend left, "\phi_2"] & \\
g \arrow[r, "\gamma"] \arrow[drr, bend right, "\phi_1"'] & k \arrow[dr, dashed, "\phi"'] & \\
 & & Q
\end{tikzcd}$$

where $Q : \mathbb{E} \to \mathbb{U}$ is a univalent Kan fibration, and where ϕ_1 and ϕ_2 are classifying f and g respectively.

Model topos

Definition

[Rezk] A combinatorial model category $\mathcal{E}$ is said to be a **model topos** if it is Quillen equivalent to a left exact Bousfield localisation of a model category $[C, \Delta Set]$ equipped with the projective model structure.

Theorem

[Rezk] *A combinatorial model category $\mathcal{E}$ is a model topos if and only if the following two conditions hold:*

- *The base change functor $f^\star : \mathcal{E}/B \to \mathcal{E}/A$ preserves homotopy colimits for every map between fibrant objects $f : A \to B$;*
- *The descent principle holds (for cubes).*

Grothendieck topos

Definition

A category $\mathcal{E}$ is said to be a **topos** if it is equivalent to a left exact localisation of a presheaf category $[C, Set]$

Theorem

A locally presentable category $\mathcal{E}$ is a topos if and only if the following two conditions hold:

- *The base change functor $f^\star : \mathcal{E}/B \to \mathcal{E}/A$ preserves colimits for every map $f : A \to B$;*
- *The presheaf $Sub : \mathcal{E}^{op} \to Set$ takes pushout squares to pullback squares.*

Grothendieck topos versus ∞-topos

Grothendieck topos	∞-topos
monomorphisms	κ-small maps
$t : 1 \to \Omega$	$\mathbb{E}_\kappa \to \mathbb{U}_\kappa$

From type theory to higher toposes

$$\begin{array}{ccc}
\text{ML type theory} & & \text{Cubical type theory} \\
\Big\downarrow{\scriptstyle \textit{syntaxic category}} & & \Big\downarrow{\scriptstyle \textit{syntaxic category}} \\
\text{tribes} & & \text{Cubical tribes} \\
\Big\downarrow{\scriptstyle \textit{localisation}} & & \Big\downarrow{\scriptstyle \textit{localisation}} \\
\text{lcc quasicategories} & & \infty - \text{toposes}
\end{array}$$

- syntaxic category [Gambino & Garner]
- localisation [Kapulkin & Szumilo]
- elementary ∞-topos [Lurie, Shulman]

What is an elementary ∞-topos ?

Tentative answer:

Definition

A tribe $\mathcal{E}$ is an **elementary infinity-topos** if

- $\mathcal{E}$ has internal products;
- $\mathcal{E}$ has inductive types;
- $\mathcal{E}$ is univalent.

An elementary infinity-topos can also be defined to be a quasi-category satisfying certain conditions. See: *A theory of elementary higher toposes* [Nima Rasekh].

Applications of type theory

- The Blakers-Massey Theorem
- The Generalised Blakers-Massey Theorem
- Goodwillie's Calculus
- Weiss's Calculus

A new proof of the Blakers-Massey theorem was found in type theory by [Finster-Licata-Lumsdaine].

The new proof was reformulated in the language of model categories by Charles Rezk.

Ref: *A mechanization of the Blakers-Massey connectivity theorem in Homotopy Type Theory* [Favonia-Finster-Licata-Lumsdaine]

The Blakers-Massey theorem

Recall that a simplical set X is said to be (-1)-*connected* if it is non-empty.

If $n \geq 0$, X is said to be n-*connected* if it is connected and $\pi_k(X, x) = 0$ for every $1 \leq k \leq n$ and $x \in X$.

Definition

We shall say that a map of simplicial set $f : X \to Y$ is n-*connected* if its homotopy fibers are n-connected.

Warning: the notion n-connected map defined above is often said to be $(n+1)$-connected in the literature.

The Blakers-Massey theorem

Theorem

(Blakers-Massey) *Suppose that we have a homotopy pushout square of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

in which f is m-connected and g is n-connected. Then the canonical map $(f, g) : A \to B \times_D^h C$ to the homotopy pullback is $(m + n)$-connected.

The Freudenthal suspension theorem

The BM theorem implies the Freudenthal suspension theorem:

If X is a pointed n-connected space, then the canonical map $X \to \Omega\Sigma X$ is $2n$-connected.

Proof: If CX is the (reduced) cone on X, then we have a pushout square

$$\begin{array}{ccc} X & \xrightarrow{n-con} & CX \\ {\scriptstyle n-con}\downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

Moreover,

$$\Omega\Sigma X \simeq CX \times_{\Sigma X} CX$$

Hence the map $X \to \Omega\Sigma X$ is $2n$-connected by the BM theorem.

The generalised Blakers-Massey theorem

The class of n-connected maps is replaced by the left class $\mathcal{L}$ of a modality.

Ref: *Modalities in homotopy type theory* [Rijke-Shulman-Spitter].

By definition, a **modality** in an ∞-topos $\mathcal{E}$ is a homotopy factorisation system $(\mathcal{L}, \mathcal{R})$ in which the left class $\mathcal{L}$ is closed under base changes.

For example, $\mathcal{L}$ can be the class of effective epimorphisms in $\mathcal{E}$ and $\mathcal{R}$ is the class of monomorphisms.

More generally, $\mathcal{L}$ can be the class of n-connected maps in $\mathcal{E}$ and $\mathcal{R}$ the class of n-truncated maps.

The generalised Blakers-Massey Theorem

Recall that the **pushout product** $f \Box g$ of two maps $f : A' \to A$ and $g : B' \to B$ in an ∞-topos $\mathcal{E}$ is the map

$$f \Box g : (A' \times B) \sqcup_{A' \times B'} (A \times B') \to A \times B$$

obtained from the commutative square

$$\begin{tikzcd} A' \times B' \arrow[r, "f \times B'"] \arrow[d, "A' \times g"'] & A \times B' \arrow[d, "A \times g"] \\ A' \times B \arrow[r, "f \times B"] & A \times B \end{tikzcd}$$

The generalised Blakers-Massey Theorem

Recall that the *diagonal* of a map $f : A \to B$ in an ∞-topos $\mathcal{E}$ is the map

$$\Delta(f) : A \to A \times_B A$$

Theorem

(GBM theorem) [ABFJ] *Let $(\mathcal{L}, \mathcal{R})$ be a modality in an ∞-topos $\mathcal{E}$ and let*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] & W \end{tikzcd}$$

be a pushout square in $\mathcal{E}$. If $\Delta(f)\Box\Delta(g) \in \mathcal{L}$, then the canonical map $(f, g) : Z \to X \times_W Y$ belongs to $\mathcal{L}$.

Ref: *The Generalised Blakers-Massey Theorem* [Anel-Biedermann-Finster-Joyal]

GBMT $\Rightarrow$ BMT

Remarks

- if f is m-connected, then $\Delta(f)$ is $(m-1)$ connected;
- If f is m-connected and g is n-connected, then $f \square g$ is $(m+n+2)$ connected.

Thus, if f is m-connected and g is n-connected, then $\Delta(f) \square \Delta(g)$ is $(m+n)$ connected.

Therefore: $GBMT \Rightarrow BMT$.

Descent $\Rightarrow$ GBMT

The proof of the GBMT depends on the following descent lemma.

Lemma

[ABFJ] *Let* $(\mathcal{L}, \mathcal{R})$ *be a modality in an* ∞*-topos* $\mathcal{E}$ *and let*

$$\begin{tikzcd} f \arrow[r, "\beta"] \arrow[d, "\alpha"'] & h \arrow[d, "\delta"] \\ g \arrow[r, "\gamma"] & k \end{tikzcd}$$

be a pushout square in $\mathcal{E}^{[1]}$. *If the square* α *and* β *are* $\mathcal{L}$*-cartesian, then so are the squares* δ *and* γ.

Goodwillie's Calculus

Let S be the quasicategory of spaces and let $\mathsf{S}_\bullet$ be the quasicategory of *pointed* spaces.

A functor $F : \mathsf{S}_\bullet \to \mathsf{S}$ is said to be a *homotopy functor* if it preserves directed colimits.

Recall that Goodwillie's Calculus associates to a homotopy functor $F : \mathsf{S}_\bullet \to \mathsf{S}$ a tower of approximations by *n-excisive* functors:

$$P_0(F) \longleftarrow P_1(F) \longleftarrow P_2(F) \longleftarrow \cdot \tag{5}$$

The first approximation $P_0(F)$ is the constant functor with value $F(0)$.

Postnikov's tower

There is an analogy between the Goodwillie tower of a functor F and the Postnikov tower of a space X

$$S_0(X) \longleftarrow S_1(X) \longleftarrow S_2(X) \longleftarrow \cdot \qquad (6)$$

The first approximation $S_0(X)$ is the set $\pi_0(X)$ of connected components of X.

The classical Blakers-Massey theorem is a powerfull tool for studying the Postnikov tower.

Biedermann's question: Is there a Blakers-Massey theorem for Goodwillie's Calculus?

Goodwillie's topos

A homotopy functor $F : \mathsf{S}_\bullet \to \mathsf{S}$ is entirely determined by its restriction to the sub-quasicategory of *finite pointed* spaces $Fin_\bullet \subset \mathsf{S}_\bullet$.

It follows that the quasi-category of homotopy functors $\mathsf{S}_\bullet \to \mathsf{S}$ is equivalent to the quasi-category $[Fin_\bullet, \mathsf{S}]$ of all functors $Fin_\bullet \to \mathsf{S}$.

The functor $P_n : [Fin_\bullet, \mathsf{S}] \to [Fin_\bullet, \mathsf{S}]$ is a left exact reflection by a theorem of Goodwillie.

(Biedermann, Rezk) The quasicategory $[Fin_\bullet, \mathsf{S}]$ is an ∞-topos, since it is a presheaf category. Hence the quasi-category $[Fin_\bullet, \mathsf{S}]_{(n)}$ of n-excisive functors is an ∞-topos.

P-equivalences

Let $\mathcal{E}$ be an ∞-topos and $P : \mathcal{E} \to \mathcal{E}$ a left exact reflection.

A map $f : X \to Y$ in $\mathcal{E}$ is said to be a P-**equivalence** if the map $P(f) : P(X) \to P(Y)$ is an equivalence.

A map $f : X \to Y$ is said to be a P-**local** if the naturality square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & P(X) \arrow[d, "P(f)"] \\ Y \arrow[r, "\eta_Y"] & P(Y) \end{tikzcd}$$

is cartesian.

This defines a modality $(\mathcal{L}_P, \mathcal{R}_P)$, where $\mathcal{L}_P$ is the class of P-equivalences and $\mathcal{R}_P$ is the class of P-local maps.

The modality $(\mathcal{L}_P, \mathcal{R}_P)$ is said to be *left exact*.

The analogy

spaces	homotopy functors
Postnikov tower	Goodwillie Tower
n-connected maps	P_n-equivalences
n-truncated maps	P_n-local maps
BM theorem	BM theorem for GC

BM theorem for Goodwillie calculus

Theorem

[ABFJ] *Consider a homotopy pushout square of homotopy functors:*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "f"'] & H \arrow[d] \\ G \arrow[r] & K \end{tikzcd}$$

If f is a P_m-equivalence and g is a P_n-equivalence, then the induced map $(f, g) : Z \to G \times_K^h H$ is a P_{m+n+1}-equivalence.

Ref: *Goodwillie's calculus of homotopy functors and higher topos theory* [Anel-Biedermann-Finster-Joyal]

Some applications

A homotopy functor F is said to be *n-homogenous* if $P_n(F) = F$ and $P_{n-1}F = P_0F$. The space $P_0(F) = F(0)$ is the *base* of F.

Theorem

(Goodwillie) *The category of n-homogenous homotopy functors over a fixed base is stable for* $n \geq 1$*.*

A homotopy functor F is said to be *n-reduced* if $P_n(F) = 0$.

Theorem

(Arone-Dwyer-Lesh) *If a homotopy functor F is n-reduced and $(2n-1)$-excisive, then it is infinitely deloopable.*

Theorem

(Goodwillie) *If $n \geq 1$ and F is a n-excisive functor with base $A := F(0)$, then the map $F \to P_{n-1}F$ is a principal G-fibration, where G is a n-homogenous functor over A.*

Vladimir Voevodsky was a visionary.

Vladimir Voevodsky was a visionary.

I hope his dream of univalent foundation will be realised in a near future!

Thank you for your attention!