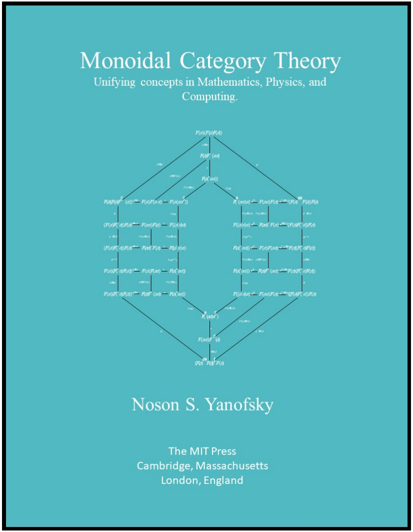
Monoidal Category Theory: Unifying concepts in Mathematics, Physics, and Computing



Chapter 1: Introduction

Foreshadowing

- Chapter : Introduction
  - Section 1.1: Categories
  - Section 1.2: Monoidal Categories
  - Section 1.3: The Examples and the Mini-courses
  - Section 1.4: Mini-course: Sets and Categorical Thinking

Foreshadowing

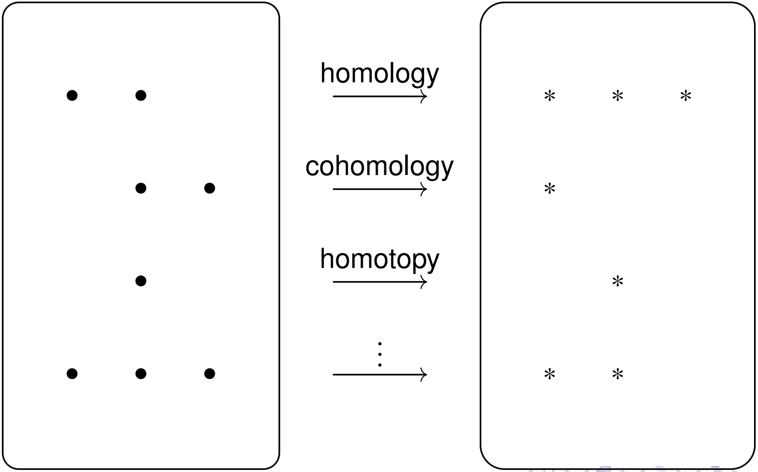
- Chapter 1: Introduction
  - Section 1.2: Categories
    - We give some historical and philosophical motivation for the concept of a category.

Some History

- Category theory began with the intention of relating and unifying two different areas of study.
- The aim was to characterize and classify certain types of geometric objects by assigning to each of them certain types of algebraic objects.
- The geometric objects are structures called topological spaces, manifolds, bundles, etc.
- The algebraic objects are called groups, rings, abelian groups, etc.
- The assignments have exotic names like homology, cohomology, homotopy and K-theory, etc.)

Some History

Geometric Objects **Algebraic Objects**



Some History

- People realized that if they were going to relate geometric objects with algebraic objects they needed a language that is neither specialized to a geometric content nor an algebraic content.
- Only with such a general language can one discuss both fields.
- This is the birth of category theory.
- It is a language about nothing, therefore it is a language about everything.

Some History

Category theory was invented by Samuel Eilenberg and Saunders Mac Lane.

Samuel Eilenberg (1913-1998)

Saunders Mac Lane (1909-2005)

Saunders Mac Lane and Samuel Eilenberg in their older years.

The first paper on category theory.

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

CONTENTS

Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual"

Presented to the Society, September 8, 1942; received by the editors May 15, 1945.

Some History

- Eilenberg and Mac Lane described various collections of mathematical objects. Each collection was called a **category**.
- There was a collection of geometric objects and a collection of algebraic objects.
- They were interested in many different categories and in order to relate one category with another, they formulated the notion of a **functor** which — like a function — assigns to each entity in one category an entity in another category.
- They went further and formulated the notion of a **natural transformation** which is a way of relating one functor to another functor. (In a sense, a natural transformation *transfers* the results of one functor to the results of another functor.)

The Three Levels of Category Theory

- There is category $\mathbb{A}$.

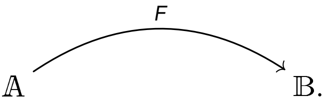
$\mathbb{A}$.

The Three Levels of Category Theory

- There is category $\mathbb{A}$.
- And category $\mathbb{B}$.

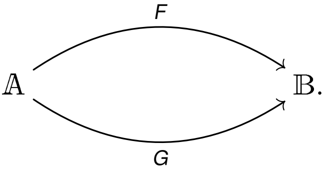
$\mathbb{A}$ $\mathbb{B}$.

The Three Levels of Category Theory



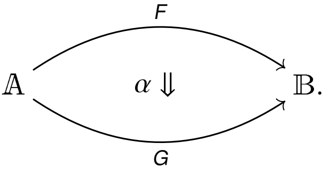
- There is category $\mathbb{A}$.
- And category $\mathbb{B}$.
- These categories are related by functor F

The Three Levels of Category Theory



- There is category $\mathbb{A}$.
- And category $\mathbb{B}$.
- These categories are related by functor F
- And by functor G.

The Three Levels of Category Theory



- There is category $\mathbb{A}$.
- And category $\mathbb{B}$.
- These categories are related by functor F
- And by functor G.
- These functors are related by natural transformation α.

Intuition

- What is a category?
- It is a collection of structures called **objects** of a particular type,
- and a collection of transformations or processes between the objects called **morphisms** or **maps**.
- If a and b are objects and f is a morphism from a to b we write it as $f\colon a \longrightarrow b$ or $a \xrightarrow{f} b$.

Intuition

- The morphisms are to be thought of as ways of transforming objects.
- As time went on, the morphisms between objects took central stage.
- Category theory became not only the study of structures but also the study of transformations or processes between structures.

Intuition

- One of the main properties of processes is that they can be combined.
- That is, one process followed by another process can be combined into a single process.

Intuition

$$a \xrightarrow{f} b$$

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Intuition

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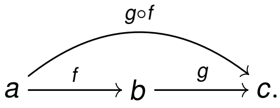
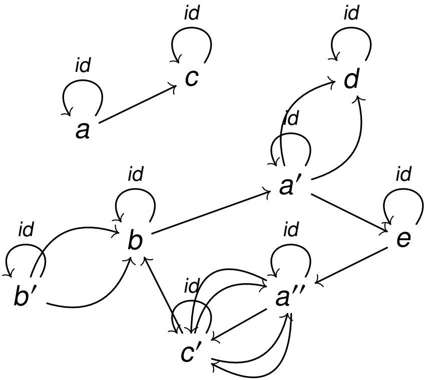
Intuition



- One of the main properties of processes is that they can be combined.
- That is, one process followed by another process can be combined into a single process.
- In a category, if there is a morphism from object a to object b called f
- and a morphism from object b to object c called g,
- then there exists an associated morphism from object a to object c written as $g \circ f$ and called "g composed f," or "g following f," or "g after f."
- This composition is the most fundamental part of a category.

Intuition

An example of part of a category.



Every object has an identity morphism (*id*).

Motivation

- Since categories are disassociated from any specific field or area, category theory received the reputation of being a language without content.
- The field was derided by some as “general abstract nonsense.”
- It is precisely this independence from any field which gives category theory its power.
- By not being formulated for one particular field, it is capable of dealing with *any* field.

Motivation

- At first, category theory was extremely successful in dealing with various fields of mathematics.
- As time went on, researchers realized that many branches of science that deal with structures or processes can be discussed in the language of category theory.
- Computer science is the study of computational processes, and hence, has taken a deep interest in category theory.
- More recently, category theory has been shown to be very adept at discussing structures and processes in physics.
- Researchers have also shown that category theory is great at discussing the structures and processes of chemistry, biology, artificial intelligence, and linguistics.

Motivation

- Many diverse fields are shown to be related because they are discussed in the single language of category theory.
- Researchers have found similar theorems and patterns in areas that were thought to be unrelated.
- Moreover, in the past few decades, category theory has further unified different fields by revealing amazing relationships between them.
- There are functors from a category in one field to a category in a totally different field that preserve properties and structures.
- Such property-preserving functors show that the two fields are similar.

Motivation

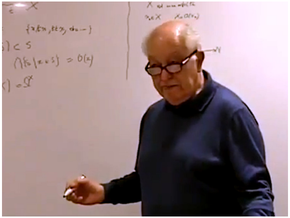
- For example, quantum algebra is a field that uses categorical language to show how certain algebraic structures are related to geometric structures like knot theory.
- Another prominent example is topological quantum field theory, which is a branch of math and physics that uses functors to unite relativity and quantum theory.
- Quantum computing is a field that sits at the intersection of computer science, physics, and mathematics and can easily be understood using various categorical structures.

Foreshadowing

- Chapter 1: Introduction
  - Section 1.2: Monoidal Categories
    - We give some historical and philosophical motivation for the concept of a monoidal category.

Some History

In the early 1960s, Jean Bénabou and Saunders Mac Lane described categories that have more structure called **monoidal categories** or **tensor categories**.



Jean Bénabou (1932-2022)

Saunders Mac Lane (1909-2005)

Monoidal Categories

- In these categories, one can "multiply" or "combine" objects.
- Symbolically, within a monoidal category, object a and object b can be combined to form object $a \otimes b$ (read "a tensor b").
- As always in category theory, one is interested not only in combining objects but also in combining morphisms.
- For $f \colon a \longrightarrow a'$ and $g \colon b \longrightarrow b'$, there exists a morphism $f \otimes g$ which we write as

$$\begin{array}{ccc} a & \xrightarrow{\quad f \quad} & a' \\ & \otimes & \\ b & \xrightarrow[\quad g \quad]{} & b' \end{array}$$

or

$$a \otimes b \xrightarrow{f \otimes g} a' \otimes b'.$$

Monoidal Categories - Foreshadowing

- Notice that there are two ways of combining morphisms in a monoidal category.
- There is $f \circ g$ and there is $f \otimes g$.
- In physics, the combination $f \circ g$ corresponds to performing one process after another while the combination $f \otimes g$ corresponds to performing two independent processes.
- In computers, the combination $f \circ g$ corresponds to sequential processes, while $f \otimes g$ corresponds to parallel processes.
- In mathematics, the interplay of the two combinations of morphisms is very important.

Monoidal Categories - Foreshadowing

- Classical algebra is the branch of mathematics that deals with sets and operations on those sets.
- For sets of numbers and the addition operation, we have the rule that $x + y = y + x$ while in general, for subtraction, $x - y \neq y - x$.
- In the theory of monoidal categories there are rules that govern the relationship between $a \otimes b$ and $b \otimes a$.
- What about the relationship between $(a \otimes b) \otimes c$ and $a \otimes (b \otimes c)$?
- Within monoidal categories there are many possible relationships when dealing with these combined objects.
- For each rule relating these operations, there will be a corresponding type of monoidal category.
- In Chapter 7, we will see many different types of monoidal categories.

Monoidal Categories - Foreshadowing

- This variability allows for many phenomena to be modeled by monoidal categories.
- The area that deals with the different types of rules among operations is called **coherence theory**
- How the various operations *cohere* with each other.
- Also called **higher-dimensional algebra**.
- This area of study has become pervasive. Higher-dimensional algebra will arise even more frequently in the science and mathematics of the coming decades.

Foreshadowing

- Chapter I: Introduction
  - Section 1.3: The Examples and the Mini-courses
    - We describe how the examples and the mini-courses are structured.
    - The way the material is spread through the book.

Organization of the Book

- This text is centered on the examples. Our goal is to show the pervasiveness of categories, and in particular, monoidal categories.
- We also want to emphasize how categories can reveal the interconnectedness of various fields.
- We do so by introducing many examples from many different areas.
- The literature of category theory has many more examples.
- We are showing the beauty of category theory but only revealing the tip of the iceberg.

Organization - Table of Contents

The examples are spread throughout the book. For example:

- In Chapter 2, a category will be introduced.
- In Chapter 3, some properties of this category will be described.
- In Chapter 4, this category will be related to other categories.
- In Chapter 5, we will show that the category has a monoidal structure.
- In Chapter 6, we will see how that monoidal structure relates to the monoidal structure of other categories.
- In Chapter 7, this same category will be shown to have even more structure .

We will also see how this category arises in various mini-courses. By the time the reader finishes the book, the category will be an old friend.

Organization - Table of Contents

One of the highlights of the book are the mini-courses.

Organization - Notation

In order to improve readability, for the most part, we keep to the following notation.

- Categories are in blackboard bold font: $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{C}\mathbb{i}\mathbb{r}\mathbb{c}\mathbb{u}\mathbb{i}\mathbb{t}, \mathbb{S}\mathbb{e}\mathbb{t}, \ldots$
- Objects in general categories are the first few lowercase Latin letters: $a, b, c, d, a', b', a'' \ldots$
- Morphisms in general categories are lowercase Latin letters: $f, g, h, i, j, k, f', g'', \ldots$
- Functors are capital Latin letters: $F, G, H, I, J, \ldots$
- Natural transformations are lowercase Greek letters: $\alpha, \beta, \gamma, \delta, \eta, \kappa, \ldots$
- Higher-dimensional morphisms will be capital Greek letters: $\Gamma, \Delta, \Theta, \Phi, \Psi, \ldots$
- 2-Categories are in blackboard bold font with a line above: $\overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, \overline{\mathbb{D}}, \overline{\mathbb{C}\mathbb{a}\mathbb{t}} \ldots$
- 3-Categories are in blackboard bold font with a two lines above: $\overline{\overline{\mathbb{A}}}, \overline{\overline{\mathbb{B}}}, \overline{\overline{\mathbb{C}}}, \overline{\overline{\mathbb{D}}}, \overline{\overline{2\mathbb{C}\mathbb{a}\mathbb{t}}} \ldots$

Organization - Notation

There are several different types of arrows in this book.

- Morphism, map or functor: $\longrightarrow$
- The input and output of a function or a functor: $\longmapsto$ or $\leadsto$
- Inclusion or injection: $\hookrightarrow$
- Surjection or full functor: $\twoheadrightarrow$
- Natural transformation: $\Longrightarrow$

Foreshadowing

- Chapter 1: Introduction
  - Section 1.4: Mini-course: Sets and Categorical Thinking
    - Sets and Operations
    - Functions
    - Operations on Functions
    - Equivalence Relations
    - Graphs
    - Groups

- We introduce basic notions of sets and functions.
- We focus on functions and how they are used to tell us about the structure of the sets.
- This is fundamental to thinking about categories.
- Basic examples of structures that sets have, such as equivalence relations, graphs, and groups are mentioned.

Central Idea

- Category theory is not just a language that is capable of describing an immense amount of science and mathematics.
- Rather, it is a *new and innovative way of thinking.*
- One of the central ideas is that we define properties of objects by the way they interact with other objects.
- In order to get a feel for this, we take an in-depth look at the familiar world of sets and functions between sets.
- We show that many of the usual ideas and constructions about sets can be described with functions between sets.
- This mini-course will also be a gentle reminder of many concepts that are needed in the rest of the course.

Important Categorical Idea

Morphisms Are Central.

- *Properties and structures in a category can be described by the morphisms of the category.*
- *The objects do not stand alone.*
- *One must see how the objects relate to each other with morphisms.*
- *The objects have to be seen in context of the morphisms.*
- *In particular, many properties of an object* b *can be understood by looking at the collections of morphisms* $a \longrightarrow b$ *for various simple objects* a.
- *Similarly, many properties of* b *are described by looking at collections of morphisms* $b \longrightarrow c$ *for various simple objects* c.

Sets - Basic Concepts

Definition

A **set** *is a collection of elements. If S is a set and x is an element of S, we write $x \in S$. If x is not an element of S we write $x \notin S$.*

Example

We will deal with both infinite sets and finite sets. Some of the most important infinite sets of numbers are

- *The natural numbers,* $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$.
- *The integers,* $\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- *The rational numbers,* $\mathbf{Q} = \{\frac{m}{n} : m \text{ and } n \text{ in } \mathbf{Z} \text{ and } n \neq 0\}$.
- *The real numbers,* $\mathbf{R}$*, that is, all numbers on the real number line.*
- *The complex numbers,* $\mathbf{C} = \{a + bi : a \text{ and } b \text{ in } \mathbf{R}\}$.

Definition

We begin by discussing several operations on sets. Let S and T be sets. If s is in S and t is in T, we write an **ordered pair** *of the elements as (s, t). The set of all ordered pairs is called the* **Cartesian product of sets** *S and T*

$$S \times T = \{(s, t) : s \in S, t \in T\}.$$

Sets - Basic Operations

Example

If Pants $=$ {*black, blue1, blue2, gray*} *is the set of pants that you own, and Shirts* $=$ {*white, blue, orange*} *is the set of shirts that you own, then the set of Outfits is*

$$Pants \times Shirts = \left\{ \begin{array}{c} (black, white), (black, blue), (black, orange), \\ (blue1, white), (blue1, blue), (blue1, orange), \\ (blue2, white), (blue2, blue), (blue2, orange), \\ (gray, white), (gray, blue), (gray, orange) \end{array} \right\}$$

Sets - Basic Operations

Technical Point

- *The most important aspect of an ordered pair is its order.*
- *In contrast, sets are just collections, and as such, do not have a preferred order.*
- *The set* $\{s, t\}$ *is considered to be the same set as* $\{t, s\}$.
- *In contrast, the pair* (s, t) *is not considered the same as* (t, s).
- *Hence, we cannot simply use two curly brackets to describe ordered pairs.*
- *There are other ways of describing an ordered pair of elements from* S *and* T*. For example:*
- *We could write them as* $\langle s, t\rangle$ *or* $\{s, t, \{s\}\}$ *, or* $\{s, t, \{t\}\}$.
- *There is nothing special about the notation* (s, t).

Definition

The most interesting property about a finite set is the number of elements in such a set. For every finite set S*, we write* $|S|$ *to denote the number of elements in* S*. If there are* m *elements in* S *and* n *elements in* T*, then there are* mn *elements in* $S \times T$*. In symbols we write this as*

$$|S \times T| = |S| \cdot |T|.$$

Definition

- *We can generalize the notion of ordered pairs to* **ordered triples**, **ordered 4-tuples**, **ordered 5-tuples**, *etc.*
- *If there are n sets, $S_1, S_2, \ldots, S_n$, then an* **ordered n-tuple** *is written as $(s_1, s_2, \ldots, s_n)$ where s_i is in S_i.*
- *The set of all n-tuples is $S_1 \times S_2 \times \cdots \times S_n$.*
- *The number of n-tuples is given as follows:*

$$|S_1 \times S_2 \times \cdots \times S_n| = |S_1| \cdot |S_2| \cdots |S_n|.$$

Sets - Basic Operations

Definition

- *Another operation performed on sets is the union.*
- *Let S and T be sets. The* **union** *of S and T is the set $S \bigcup T$ which contains those elements that are in S or in T.*

$$S \bigcup T = \{x : x \in S \text{ or } x \in T\}.$$

- *It is important to notice that if there is some element that is in both S and in T then it will occur only once in $S \bigcup T$.*
- *This is because when dealing with a set, repetition does not matter. The set $\{a, b, c, b\}$ is considered the same set as $\{a, b, c\}$.*

Sets - Basic Operations

Definition

- *A related operation is the disjoint union.*
- *Given sets S and T, one forms the* **disjoint union** $S \amalg T$ *which contains the elements from S and T but considers elements that are in both sets as different elements.*
- *One way this is done is by tagging every element with extra information that says which set it comes from. This way an element that is both in S and in T would be considered two different elements.*

Sets - Basic Operations

Example

- *For example, if* $S = \{a, b, c, x, y\}$ *and* $T = \{q, w, b, x, e, r\}$*, then* $S \amalg T$ *is*

$$\{(a,0),(b,0),(c,0),(x,0),(y,0),(q,1),(w,1),$$

$$(b,1),(x,1),(e,1),(r,1)\},$$

where the elements of S *are tagged with a* 0 *and the elements of* T *are tagged with a* 1*.*

- *In general, for sets* S *and* T*, we have*

$$S \amalg T = (S \times \{0\}) \bigcup (T \times \{1\}).$$

- *The formula for the number of elements in the disjoint union is*

$$|S \amalg T| = |S| + |T|.$$

Functions - Basic Definitions

The central idea of this mini-course is the notion of functions between sets and how they determine properties of sets.

Definition

Let S and T be sets. A **function** *f from S to T, written $f\colon S \longrightarrow T$ is an assignment to every element of S an element of T. The value of f on the element s is written as $f(s)$ ("f of s"). If $f(s) = t$ we write $s \mapsto t$ ("s maps to t").*

- It is important to understand the difference between the symbol $\longrightarrow$ and the symbol $\mapsto$.
- The symbol $\longrightarrow$ goes between two sets. It describes a function from one set to another.
- In contrast, the symbol $\mapsto$ goes from an element in the first set to an element in the second set. It describes how the function is defined.

Functions - Basic Definitions

Example

For every set S, there is an **identity function** $id_S : S \longrightarrow S$ *which takes every element to itself. In symbols it is defined as* $id_S(s) = s$ *or* $s \mapsto s$.

Important Categorical Idea

Not Entities, But Morphisms Between Entities.

- *In category theory, whenever we have a notion (for example, a set), the immediate next task is to consider how these notions relate to each other.*
- *Category theory is not about "things," but about how "things" relate to "things."*
- *Relations between objects are usually described by morphisms or functions between the objects.*
- *Following this rule, once we describe the morphisms between the objects we must immediately ask what is between the morphisms. Usually the answer will be other morphisms.*
- *The computer scientist might protest that this recursive procedure will lead into an infinite loop. It will! We will see this in higher-dimensional category theory.*

Functions - Basic Definitions

- Let us look at how morphisms to a set determine properties of a set.
- Functions can be used as a way of describing or choosing elements of a set.
- Consider a one-element set, $\{*\}$. (There are many one-element sets such as $\{a\}$, $\{b\}$, $\{\text{Bill}\}$, etc.)
- For a set S, a function $f\colon \{*\} \longrightarrow S$ picks out one element of S.
- The single element $*$ goes to the selected element s in S.
- In symbols, $f(*) = s$ or $* \mapsto s$.

Example

- *Let S be the set* $\{$*Jack, Jill, Joan, June, Joe, John*$\}$.
- *The element Joe in S can be described as a function* $f\colon \{*\} \longrightarrow S$ *where* $f(*) =$ *Joe. We might want to distinguish this function by calling it* $f_{Joe}\colon \{*\} \longrightarrow S$.
- *There will be other functions like* $f_{Jill}\colon \{*\} \longrightarrow S$ *where* $f_{Jill}(*) =$ *Jill.*
- *For this set of six elements, there are six different functions from* $\{*\}$ *to* S.

Functions - Examples

Example

- *If we are interested in choosing two elements of S, we can look at functions from a two-element set to S.*
- *So $f: \{0, 1\} \longrightarrow S$ will choose two elements of S. The first element is $f(0)$ and the second element is $f(1)$.*
- *If $f(0) \neq f(1)$ then f will choose two different elements of S.*
- *Functions from $\{a, b, c\}$ to S will choose three elements of S.*
- *If we wanted to choose n elements of a set, we would look at functions of the form $\{1, 2, \ldots, n\} \longrightarrow S$.*

Functions - Examples

Definition

- *Set T is a* **subset** *of S if every element of T is an element of S.*
- *We write this as $T \subseteq S$.*
- *If T is a subset of S but not equal to S, we call T a* **proper subset** *and write $T \subsetneq S$ or $T \subset S$. This is the case when there is at least one element in S that is not in T.*
- *If T is a subset of S, there is an* **inclusion function** *that takes every element of T to its corresponding element of S which is written as $inc\colon\ T \hookrightarrow S$.*
- *There is a special set that has no elements called the* **empty set** *and denoted $\emptyset$. Since it is true that whatever is in $\emptyset$ (nothing) is in any other set, we have that the empty set is a subset of every set. Furthermore, for every set S, there is a unique function from the empty set to S. We sometimes denote this function as $!\colon \emptyset \longrightarrow S$.*

Functions - Examples

Example

- *Subsets of a set are of fundamental importance. We shall be interested in the collection of all subsets of a particular set*
- *For a set S, the set of all subsets of S is called the* **power set** *of S and is denoted $\mathcal{P}(S)$. In other words,*

$$\mathcal{P}(S) = \{T : T \text{ is a subset of } S\}.$$

- *For the set $\{a\}$, the powerset is $\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$.*
- *The powerset of a two element set is $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.*
- *The powerset of a three-element set is $\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.*
- *Whenever we add an element to a set, we double the number of elements in the powerset. We have following rule: $|\mathcal{P}(S)| = 2^{|S|}$.*

Functions - Examples

Functions can be used to describe subsets.

Definition

For any set S and subset $T \subseteq S$, there is an associated **characteristic function** $\chi_T \colon S \longrightarrow \{0, 1\}$*. This function assigns either 1 or 0 to every element s of S. If s is in T, the characteristic function assigns a 1 to s, and if s is not in T, it assigns a 0 to s, i.e.,*

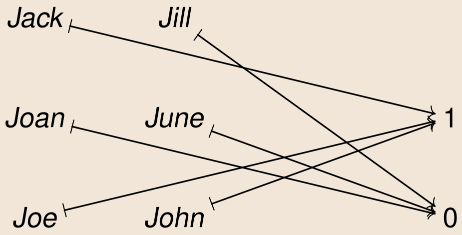
$$\chi_T(s) = \begin{cases} 1 & : s \in T \\ 0 & : s \notin T. \end{cases}$$

(The Greek letter χ is pronounced "chi" and is supposed to remind you of the first syllable of "characteristic".) The function χ_T tells which elements of S are in T and which elements of S are not in T. Characteristic functions establish a correspondence between subsets of S and functions from S to $\{0, 1\}$.

Functions - Examples

Example

Let S be the set {Jack, Jill, Joan, June, Joe, John}. Consider the subset $T = \{$Jack, Joe, John$\}$ of S that contains all the boys in S. This subset can be described by the function $\chi_T : S \longrightarrow \{0, 1\}$ which can be visualized as



Functions - Examples

Example

- *A characteristic function assigns the elements of S to one of two possible values.*
- *There might be a need to assign one of many values to every element of S.*
- *In general, a function $S \longrightarrow \{1, 2, \ldots, n\}$ assigns every element of S one of n numbers.*
- *We can also assign to every element of S an element of $[0, 1]$, the real interval between 0 and 1. Such a function may correspond to assigning a probability to every element.*
- *In school, every student usually has an associated grade point average (GPA). This is written as a function $Students \longrightarrow [0, 4]$.*

Functions - Examples

Example

- *If S is a set, then there is a function called the* **diagonal function** $\Delta\colon S \longrightarrow S \times S$ *which takes every element to an ordered pair of the same element. In symbols, for s in S we have*

$$\Delta(s) = (s, s).$$

- *If $f\colon S \longrightarrow S'$ and $g\colon T \longrightarrow T'$ are functions then there exists a function $f \times g\colon S \times T \longrightarrow S' \times T'$ that takes an ordered pair of elements and applies f to the first element and g to the second. In symbols, the function is defined for elements s of S and t of T as*

$$(f \times g)((s, t)) = (f(s), g(t)) \in S' \times T'.$$

- *In a sense, this process is a parallel process. The function f processes s while the function g processes t.*

Functions - Special Types

Definition

There are some special types of functions. We say $f\colon S \longrightarrow T$ is

- **one-to-one** *or* **injective** *if different elements in S go to different elements in T. That is, for all s and s' in S, if $s \neq s'$ then $f(s) \neq f(s')$. Another way to say this is that if $f(s) = f(s')$, then it must be that $s = s'$. This means that if the function takes elements to the same output, the elements must have started off equal.*
- **onto** *or* **surjective** *if for every element t in T, there is an s in S such that $f(s) = t$.*
- *an* **isomorphism** *or a* **one-to-one correspondence** *or a* **bijection** *if f is one-to-one and onto. That is, for every element s of S there is a unique element t of T so that $f(s) = t$ and for every element t of T there is a unique element s of S so that $f(s) = t$.*

Function Set - Example

Definition

- *One of the central ideas about sets is that given sets S and T we can form a set which consists of all functions from S to T.*
- *We call this collection a* **set of functions** *or a* **function set** *or a* **Hom set** *and we denote it as $Hom(S, T)$ or T^S. (The notation $Hom(S, T)$ comes from the word "***hom***omorphism" which is a vestige of the algebraic origins of the idea.*
- *The notation T^S is similar to exponentiation because the function set has similar properties to exponentiation.)*

Function Set - Examples

Example

Let us write down the set of all the functions from the set $\{a, b, c\}$ *to the set* $\{0, 1\}$*. Each of the following lines is a function.*

$f(a) = 0$	$f(b) = 0$	$f(c) = 0$
$f(a) = 0$	$f(b) = 0$	$f(c) = 1$
$f(a) = 0$	$f(b) = 1$	$f(c) = 0$
$f(a) = 0$	$f(b) = 1$	$f(c) = 1$
$f(a) = 1$	$f(b) = 0$	$f(c) = 0$
$f(a) = 1$	$f(b) = 0$	$f(c) = 1$
$f(a) = 1$	$f(b) = 1$	$f(c) = 0$
$f(a) = 1$	$f(b) = 1$	$f(c) = 1$

Function Set - Examples

Example

- *We saw that every element in a set S can be described as a function $\{*\} \longrightarrow S$. This correspondence between elements of S and functions from $\{*\}$ to S shows that*

$$S \cong S^{\{*\}} = Hom(\{*\}, S).$$

- *Using characteristic functions, we saw there is a correspondence between subsets of S and functions from S to $\{0, 1\}$. This correspondence is stated as*

$$\mathcal{P}(S) \cong \{0, 1\}^S = Hom(S, \{0, 1\}).$$

- *We denote the set $\{0, 1\}$ as 2 and then write this as*

$$\mathcal{P}(S) \cong 2^S = Hom(S, 2).$$

Functions - Examples

Example

- *Consider the binary addition operation* $+\colon \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$*. Let us write this function with its inputs clearly marked as follows*

$$(\ \ ) + (\ \ )\colon \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}.$$

- *Now consider the function* $(\ \ ) + 5\colon \mathbf{N} \longrightarrow \mathbf{N}$*. This is a function with only one input. We could also make another function of one variable* $(\ \ ) + 7\colon \mathbf{N} \longrightarrow \mathbf{N}$*. In fact we can do this for any natural number.*
- *We can define a function that inputs a natural number and outputs a function from natural numbers to natural numbers. That is, there is a function* $\Phi\colon \mathbf{N} \longrightarrow Hom(\mathbf{N}, \mathbf{N})$ *which is defined as* $\Phi(n) = (\ \ ) + n$*.*
- *The information described by the* $(\ \ ) + (\ \ )$ *function is the same as the information described by the* Φ *function.*

Functions - Examples

Example

- *Notice that what we said about* $+$ *really applies to every function with two inputs.*
- *If* $f\colon S \times T \longrightarrow U$ *is a function from* $S \times T$*, then for every* $t \in T$ *there is a function* $f(\ \ ,t)\colon S \longrightarrow U$*.*
- *This shows that there is a function* $f' : T \longrightarrow Hom(S, U)$*.*
- *The assignment described by any function* f *has the same information as the assignment described by the function* f'*.*
- f *and* f' *relay the same information.*

Functions - Theorem

These examples brings to light the following important theorem.

Theorem

For sets S, T and U there is an isomorphism

$$Hom(S \times T, U) \cong Hom(T, Hom(S, U)) \qquad or \qquad U^{S\times T} \cong (U^S)^T$$

Proof.

- Consider $f\colon S \times T \longrightarrow U$.
- From this function let us define an $f'\colon T \longrightarrow Hom(S, U)$.
- For a t in T we have the function $f'(t)\colon S \longrightarrow U$ which is defined as follows: for s in S, let $f'(t)(s) = f(s, t)$.
- This function has the same information as f.
- One can go from f' to f also.

□

Functions - Counting

- Let us count how many functions there are between two finite sets.
- Consider S with $|S| = m$ and T with $|T| = n$, and a function $f \colon S \longrightarrow T$.
- For each element s in S there are n possible values of $f(s)$ in T.
- For two elements in S there are $n \cdot n$ possibilities of choices in T.
- In total, there are $n \cdot n \cdot \cdots \cdot n$ (m times) possible maps. So

$$|Hom(S, T)| = |T^S| = n^m = |T|^{|S|}.$$

Remark

For three finte sets S, T, and U, we have

$$
\begin{aligned}
|Hom(S \times T, U)| &= |U^{S \times T}| \\
&= |U|^{|S \times T|}) \\
&= |U|^{|S| \cdot |T|} \\
&= (|U|^{|S|})^{|T|} \\
&= |Hom(S, U)|^{|T|} \\
&= |Hom(T, Hom(S, U))|
\end{aligned}
$$

Notice that the rule about exponentiation usually learned as children, $m^{(n \cdot p)} = (m^n)^p$ *is expanded to a rule about sets and functions.*

Operations on Functions

Often we are going to take two functions and perform an operation to get another function. Three such operations are

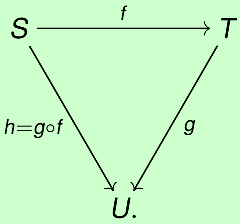
- composition,
- extension, and
- lifting.

Remarkably, many ideas about functions can be understood as operations in one of these three forms.

Operations on Functions - Composition

Definition

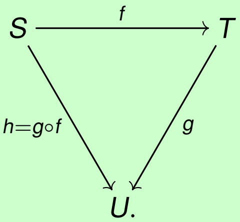
- *The simplest operation is* **composition**.
- *If there is a function* $f\colon S \longrightarrow T$ *and a function* $g\colon T \longrightarrow U$, *then the composite of them is a function* $h = g \circ f\colon S \longrightarrow U$.
- *It is defined on an* s *in* S *as* $h(s) = g(f(s))$.
- *We write these functions as*



- *We say that* f *and* g *are* **factors** *of* h *or* h **factors through** T.

Operations on Functions - Composition

Definition

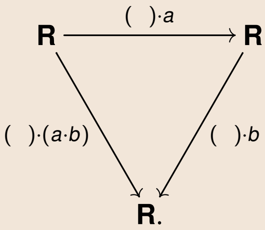


- *This diagram is called a* **commutative diagram***.*
- *If you start with any element s in S and you apply the functions f followed by g, you will get to the same resulting element as applying the function h.*
- *In detail, for all s, we have that* $g(f(s)) = h(s)$*.*
- *The diagram is "commutative" because we can go around the triangle this way or that way.*

Operations on Functions - Composition

Example

- *Let a and b be real numbers.*
- *Consider the function $(\ \ )\cdot a\colon \mathbf{R} \longrightarrow \mathbf{R}$ that takes any real number and multiplies it by a.*
- *There is also a function $(\ \ )\cdot b\colon \mathbf{R} \longrightarrow \mathbf{R}$ that multiplies by b.*
- *The composition of these two functions is the function $(\ \ )\cdot(a\cdot b)\colon \mathbf{R} \longrightarrow \mathbf{R}$ that takes any real number and multiplies it by $a \cdot b$*



Operations on Functions - Composition

Theorem

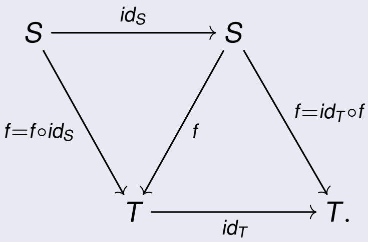
Function composition is associative. That is, let $f\colon S \longrightarrow T$*,* $g\colon T \longrightarrow U$ *and* $h\colon U \longrightarrow V$*, then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

When dealing with the identity function, the input is the same as the output. This has an interesting consequence when dealing with composition.

Theorem

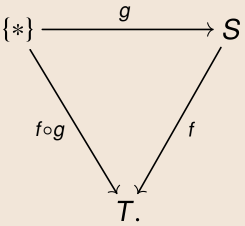
If you compose a function with an identity map, then you get the original function. In detail, for $f\colon S \longrightarrow T$, $id_S\colon S \longrightarrow S$, *and* $id_T\colon T \longrightarrow T$, *we have* $f \circ id_S = f$ *and* $id_T \circ f = f$. *We can see these equations as the following commutative diagram:*



Operations on Functions - Composition

Example

Evaluation of a function can be seen as composition. Let $f\colon S \longrightarrow T$ be a function and let an element be described by the function $g\colon \{*\} \longrightarrow S$. Then the value of f on the element that g chooses is the element that $f \circ g$ chooses, as in

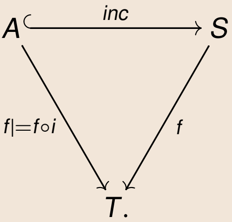


If g performs the assignment $* \mapsto s$, then $f \circ g$ performs the assignment $* \mapsto f(s)$.

Operations on Functions - Composition

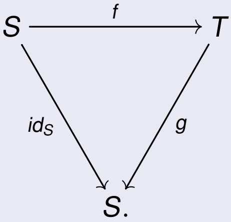
Example

- *A restriction function is an example of composition.*
- *Let* $f\colon S \longrightarrow T$ *be a function.*
- *Let* A *is a subset of* S *with inclusion function* $inc\colon A \hookrightarrow S$.
- *The restriction of* f *to* A *is the function* $f|\colon A \longrightarrow T$ *which is given as the composition*



Theorem

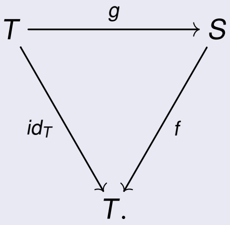
The three properties of functions that we saw can be described with function composition.
For non-empty sets S and T, the function $f\colon S \longrightarrow T$ is **one-to-one** *if and only if there exists a $g\colon T \longrightarrow S$ such that $g \circ f = id_S$*



(Proof. The existence of a g such that the diagram commutes implies f is one-to-one. If $f(s) = f(s')$ then apply g to both sides of the equation and get $g(f(s)) = g(f(s'))$. But $g \circ f = id_S$ implies $s = s'$.
If f is one-to-one, then there exists a g such that the diagram

Theorem

onto *if and only if there exists a* $g\colon T \longrightarrow S$ *such that* $f \circ g = id_T$



(Proof. The existence of a g *implies* f *is onto. The function* f *is onto because for any* $t \in T$*, the function* g *has* $g(t) = s$ *for some* $s \in S$*. This* s *gives an input to* f *whose output is* t*, i.e.,*

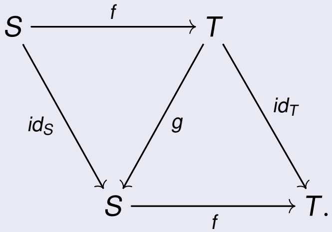
$f(s) = f(g(t)) = id_T(t) = t$.

Onto implies the existence of g *such that the diagram commutes. Let* $g(t)$ *equal any* s *such that* $f(s) = t$*. There must be one such* t *because* f *is onto. This proof assumes the axiom of choice which*

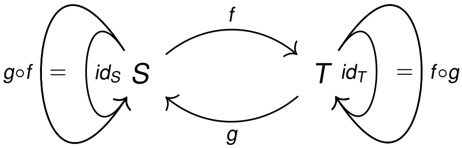
Operations on Functions - Composition

Theorem

isomorphism *or* **one-to-one correspondence** *if and only if there exists a* $g\colon T \longrightarrow S$ *such that* $g \circ f = id_S$ *and* $f \circ g = id_T$. *Or putting the previous two triangles together, we have*



Another way to express isomorphism:

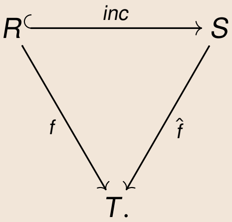


We will see variations of this diagram again and again.

Operations on Functions - Extension

Example

- *A second operation of functions is an* **extension**.
- *If* $f\colon R \longrightarrow T$ *is a function and* R *is a subset of* S *with the inclusion function* $inc\colon R \hookrightarrow S$, *then an extension of* f *along* inc *is a function* $\hat{f}\colon S \longrightarrow T$ *such that the following commutes*



- *In English,* $\hat{f}$ *extends* f *to a larger domain.*

Example

- *As a simple example, consider R to be a set of students and*

$$f\colon\ R \longrightarrow \{A, B, C, D, F\}$$

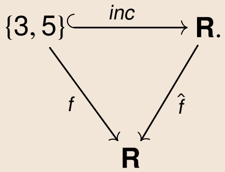
  assigns every student a grade.
- *If some new students came into the class, the teacher would have to extend f to give grades to all the students (including the new ones) as* $\hat{f}\colon\ S \longrightarrow \{A, B, C, D, F\}$.
- *We want* $\hat{f}$ *to assign the same grades as f did for any of the original students.*
- *This is clear with the following commutative diagram:*

$$\begin{array}{ccc} \{\textit{original students}\} & \overset{inc}{\hookrightarrow} & \{\textit{original and new students}\} \\ & {}_{f}\searrow \quad \{A, B, C, D, F\} \quad \swarrow_{\hat{f}} & \end{array}$$

Operations on Functions - Extension

Example

- *Let* $\{3,5\}$ *be a set of two real numbers.*
- *There is an obvious inclusion* $inc\colon \{3,5\} \hookrightarrow \mathbf{R}$.
- *Let* $f\colon \{3,5\} \longrightarrow \mathbf{R}$ *be any function that picks two values.*
- *Then, there exists a function* $\hat{f}\colon \mathbf{R} \longrightarrow \mathbf{R}$ *that extends* f.

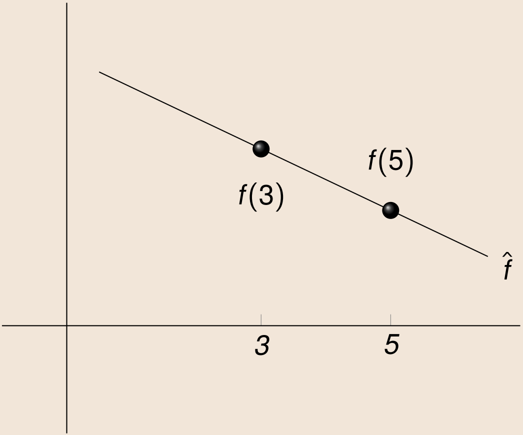


- *This extension is another way of describing the simple idea that given any two points on the plane, there is a straight line that passes through both of them.*

Operations on Functions - Extension

Example

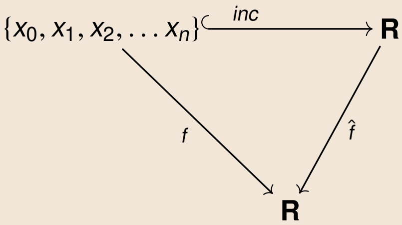
- *This can be visualized as*



- The previous example of an extension can be ... extended...

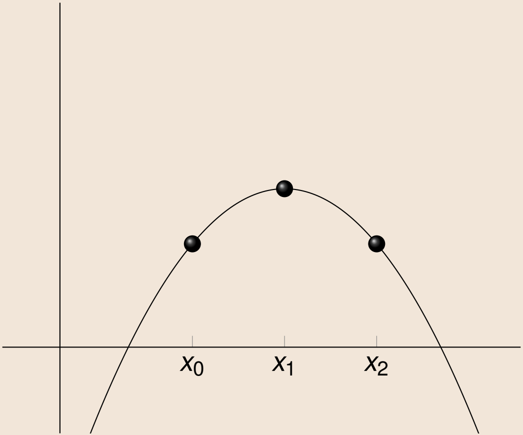
Example

Let $\{x_0, x_1, x_2, \ldots, x_n\}$ be a set of $n+1$ different real numbers and let $inc\colon \{x_0, x_1, x_2, \ldots, x_n\} \hookrightarrow \mathbf{R}$ be the inclusion function. Every $f\colon \{x_0, x_1, x_2, \ldots, x_n\} \longrightarrow \mathbf{R}$ has an extension called $\hat{f}\colon \mathbf{R} \longrightarrow \mathbf{R}$ along inc which is a polynomial function of degree at most n.



Example

This can be visualized as

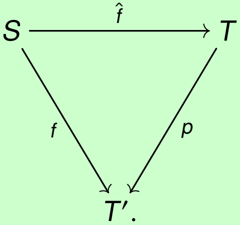


The function $\hat{f}$ is called the "Lagrange interpolating polynomial" of the points described by f. (We will not use this.)

The third operation of functions is a lifting.

Definition

Consider an onto function $p\colon T \longrightarrow T'$. Let $f\colon S \longrightarrow T'$ be any function. A **lifting** *of f along p is a function $\hat{f}\colon S \longrightarrow T$ that makes the following triangle commute*

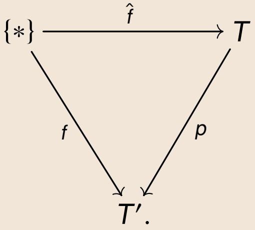


In a sense, we "lift" the map f from the target of p to the source of p.

Operations on Functions - Lifting

Example

Consider the following commutative diagram:



- *Here a function* $f\colon \{*\} \longrightarrow T'$ *picks out an element of* T'.
- *A lifting of* f *is* $\hat{f}\colon \{*\} \longrightarrow T$ *which picks out an element of* T.
- p *will take outputs of* $\hat{f}$ *to outputs of* f.
- *The set of all possible elements that a lifting can pick is denoted* $p^{-1}(t_0) \subseteq T$ *which is called the "preimage" of* p.

Operations on Functions - Lifting

Example

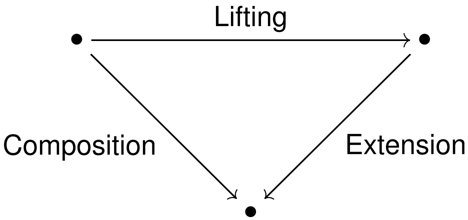
- *Here is a cute example of a lifting from the world of politics.*
- *Let T be the set of 320 million American citizens and let T' be the set of 50 states.*
- *The function p takes every citizen to the state they live in.*
- *Let S be a set of three elements such as $\{a, b, c\}$. The function $f\colon S \longrightarrow T'$ chooses three states.*
- *A lifting of f along p is a function $\hat{f}\colon S \longrightarrow T$ which will choose three citizens. Each of the three will be from the states f chooses.*
- *There are obviously many such liftings.*

Operations on Functions - Lifting

Example

- *Let us build on the last example. Let T, T' and p be as in the last example.*
- *Let S be the set $\{a, b, c\} \times T'$, i.e., pairs of letters and states.*
- *The $f\colon S \longrightarrow T'$ function is defined as follows: $f(b, \text{New Jersey}) = \text{New Jersey}$, i.e., f takes a letter and a state and outputs the same state.*
- *A lifting of f along p is a function $\hat{f}\colon S \longrightarrow T$ which will choose three citizens from each state.*

One can see these three operations — composition, extension, and lifting — as three sides of a triangle:



Each side uses the other two sides as the input to the operation. Composition will be used on almost every slide. We will see that the extension and lifting operations are very important in many contexts besides sets and functions.

Foreshadowing

The rest of this mini-course will be concerned with the following structures based on sets:

- Equivalence Relations
- Graphs
- Groups

We are not only interested in how a set is related to other sets. Sometimes the elements of a set are related to each other in interesting ways.

Definition

Let S be a set. A **relation** *on S is a subset R of the set $S \times S$. The ordered pair (s_1, s_2) in R means s_1 is related to s_2.*

Equivalence Relations

Example

Let S be the set of citizens of the United States. Consider the following relations on this set.

- R_1 *consists of those* (s, t) *where* s *and* t *are cousins.*
- R_2 *consists of those* (s, t) *where* s *is the same age or older than* t.
- R_3 *consists of those* (s, t) *where* s *and* t *live in the same state.*
- R_4 *consists of those* (s, t) *where* s *and* t *belong to the same political party.*

Equivalence Relations

The following three properties of a relation will characterize the notion of “sameness.”

Definition

The relation $R \subseteq S \times S$ on a set S is

- **reflexive** *if every element is related to itself: for all s in S, (s, s) is in R.*
- **symmetric** *whenever one element is related to another, then the other is related to the first: for all s and t in S, if (s, t) is in R, then (t, s) is in R.*
- **transitive** *whenever s is related to t and t is related to u, then s is related to u: for all s, t and u in S, if (s, t) is in R and (t, u) is in R, then (s, u) is in R.*

Equivalence Relations

Example

Let us look which properties are satisfied from the relations of Example.

- *The cousin relation* R_1 *is not reflexive (no one is their own cousin); it is symmetric; but it is not transitive (*x *can be a cousin to* y *through* y*'s mother's side and* y *can be a cousin to* z *through* y*'s father's side. In this case* x *will, in general, not be cousins to* z*.)*
- *The older relation* R_2 *is reflexive (everyone is the same age as themselves), not symmetric (if* x *is older than* y *then* y *is not older than or the same age as* x*), and it is transitive.*
- *The state relation* R_3 *is reflexive, symmetric, and transitive.*
- *The political party relation* R_4 *is reflexive, symmetric, and transitive.*

Equivalence Relations

Definition

A relation on a set is an **equivalence relation** *if it is reflexive, symmetric, and transitive. We write such relations as* $\sim \subseteq S \times S$ *and write* $r \sim s$ *for* $(r, s) \in \sim$.

Equivalence Relations

- Many times a set of elements can be split up or partitioned into different subsets where each subset will have all the elements with the “same” particular property.
- For example, the set of cars can be split up by color: there is a subset of blue cars, a subset of red cars, a subset of green cars, etc.
- The collection of all such subsets will form a set itself.

Equivalence Relations

Let us be formal.

Definition

With an equivalence relation on the set S, we can describe disjoint subsets of S called **equivalence classes**. *If s is an element of S, then the equivalence class of s is the set of all elements that are related to it:*

$$[s] = \{r \in S : r \sim s\}$$

That is, $[s]$ is the set of all elements that are "the same" as s. For a given set S and an equivalence relation $\sim$ on S, we form a **quotient set** *denoted $S/\sim$. The elements of $S/\sim$ are all the equivalence classes of elements in S. There is an obvious* **quotient function** *from S to $S/\sim$ that takes s to $[s]$.*

Equivalence Relations

Example

Let us examine the equivalence classes for the equivalence relations of Example 27.

- *Each equivalence class for the relation R_3 consists of all the residents of a particular state. The quotient set contains the 50 equivalence classes corresponding to the 50 States (we are ignoring abnormalities like Guam and Washington D.C.). The quotient function takes every citizen to the state in which they reside.*
- *Each equivalence class for the relation R_4 consists of all the people belonging to a particular political party. The quotient set is a set whose elements correspond to political parties. The quotient function takes every citizen to the political party to which they belong (we are ignoring independents.)*

Graphs

A directed graph is a common structure (based on sets) that has applications everywhere. Directed graphs also have many similarities to categories.

Definition

A **directed graph** $G = (V(G), A(G), src_G, trg_G)$ *is*

- *a set of vertices,* $V(G)$*, and*
- *a set of arrows,* $A(G)$*.*

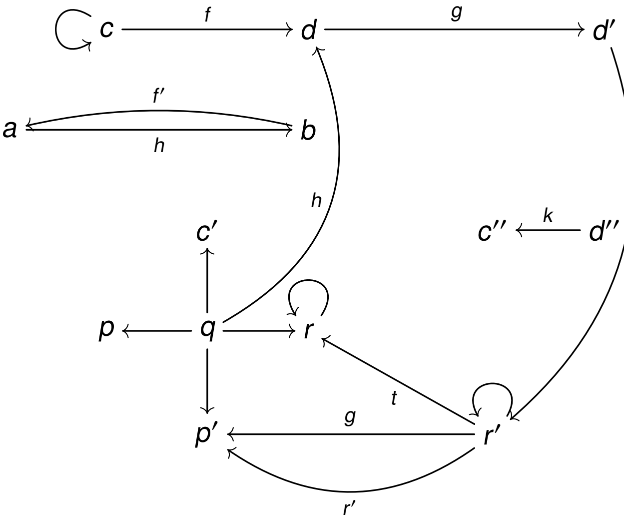
Furthermore,

- *every arrow has a source: there is a function* $src_G \colon A(G) \longrightarrow V(G)$*, and*
- *every arrow has a target: there is a function* $trg_G \colon A(G) \longrightarrow V(G)$*.*

If f *is an element of* $A(G)$ *with* $src_G(f) = x$ *and* $trg_G(f) = y$*, we draw this arrow as* $x \xrightarrow{\quad f \quad} y$ *.*

Graphs

An example of a graph:



Graphs

Example

Graphs are everywhere.

- *A street map can be thought of as a directed graph where the vertices are street corners and there is an arrow from one corner to the other if there is a one-way street between them. When there is a two-way street, we might write it like this*

  $$* \;\underset{\longleftarrow}{\overset{\longrightarrow}{}}\; * \qquad \text{or} \qquad * \;\rule{2cm}{0.4pt}\; *.$$

  Such an arrow is called a "symmetric edge."

- *An electrical circuit can be viewed as a directed graph. The vertices are the branching points and edges might have resistors, batteries, capacitors, diodes, etc. The arrows describe the direction of the flow of electricity.*

Graphs

Example

Graphs are everywhere.

- *Computer networks can be seen as directed graphs where the vertices are computers and there is an arrow from one computer to another if there is a way for the first computer to communicate with the second.*
- *The billions of web pages in the World Wide Web form a directed graph. The vertices are the web pages and there is an arrow if there is a link from one web page to another.*
- *Facebook can be seen as a directed graph. Every personal Facebook account is a vertex, and there are arrows between two Facebook accounts if they are friends. Notice that if x is friends with y then y must be friends with x. So all the arrows are symmetric edges and the graph is called symmetric.*

Graphs

Example

Graphs are everywhere.

- *All the people on Earth form a graph. The vertices are the people. There is an arrow from x to y if x knows y. (We are not being specific as to what it means to "know" someone.) There is an idea called "six degrees of separation," which says that in this graph, you never need to traverse more than six arrows to get from any person to any other person. We are all connected!*
- *The collection of all sets and functions form a giant graph. In detail, the vertices are all sets. The arrows are functions from a set to another set. (This will be a motivating example of a category.)*

A graph homomorphism is a way of mapping one graph to another. This will be similar to what happens when we talk about mapping one category to another category. Basically the vertices map to the vertices and the arrows map to the arrows but we insist that they match up well.

Definition

Let $G = (V(G), A(G), src_G, trg_G)$ *and* $G' = (V(G'), A(G'), src_{G'}, trg_{G'})$ *be graphs. A* **graph homomorphism** $H\colon G \longrightarrow G'$ *consists of*

- *A function that assigns vertices to vertices,* $H_V\colon V(G) \longrightarrow V(G')$*.*
- *A function that assigns arrows to arrows,* $H_A\colon A(G) \longrightarrow A(G')$*.*

Graphs

Definition

These two maps must respect the source and target of each arrow. That means:

- *For all f in $A(G)$, $H_V(src_G(f)) = src_{G'}(H_A(f))$.*
- *For all f in $A(G)$, $H_V(trg_G(f)) = trg_{G'}(H_A(f))$.*

Saying that these axioms are satisfied is the same as saying that the following two squares commute:

$$\begin{array}{ccc} A(G) & \xrightarrow{H_A} & A(G') \\ {\scriptstyle src_G}\downarrow & & \downarrow{\scriptstyle src_{G'}} \\ V(G) & \xrightarrow[H_V]{} & V(G') \end{array} \qquad \begin{array}{ccc} A(G) & \xrightarrow{H_A} & A(G') \\ {\scriptstyle trg_G}\downarrow & & \downarrow{\scriptstyle trg_{G'}} \\ V(G) & \xrightarrow[H_V]{} & V(G'). \end{array}$$

Definition

Another way to understand these requirements is to see what the maps H_V and H_A do to a single arrow f (that is, $f \mapsto H_A(f)$. We use the wavy arrow so that the reader can see what is going on easier.)

Graph G **Graph G′**

$$src_G(f) \overset{H_V}{\rightsquigarrow} H_V(src_G(f)) = src_{G'}(H_A(f))$$

$$f \; \overset{H_A}{\rightsquigarrow} \; H_A(f)$$

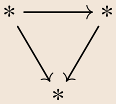
$$trg_G(f) \overset{H_V}{\rightsquigarrow} H_V(trg_G(f)) = trg_{G'}(H_A(f))$$

Graphs

Just as we can determine many properties of sets by examining functions, we can also determine many properties of graphs by examining graph homomorphisms from simple graphs.

Example

- *A vertex of a graph G can be described by a graph homomorphism from the one-vertex graph* $(*)$ *(no arrows) as follows* $H\colon * \longrightarrow G$.
- *A directed edge of a graph can be determined by a graph homomorphism from the graph* $* \longrightarrow *$ *to G.*
- *A triangle in a graph G can be determined by a graph homomorphism from the graph*



to the graph G.

Example

Graph homomorphisms can be used to determine different types of paths in a graph:

- *A simple path in a graph (a simple path is a path that does not have repeated vertices).*
- *A cycle of length n (a cycle is a path that starts and ends at the same vertex).*
- *A simple cycle of length n (a simple cycle is a cycle in which the only repeating vertex is the starting point which is the ending point).*

Theorem

The composite of graph homomorphisms is a graph homomorphism. The composition is an associative operation.

Theorem

There is an **identity graph homomorphism**, I_G, *for any graph* G. *If* $H\colon G \longrightarrow G'$ *is a graph homomorphism then* $H \circ I_G = H$ *and* $I_{G'} \circ H = H$.

Groups

Another important structure that is based on sets and related to categories is a group. It is nice to see the definition of a group from a function perspective.

First a discussion of operations.

Definition

- *We all know what we mean by operations on numbers. If you take numbers x and y, you can perform the addition operation, $x + y$ or $x - y$ or $y \cdot x$.*
- *All of these are examples of binary operations. Operations are really just functions. For a given set, S, a* **binary operation** *is a function $f\colon S \times S \longrightarrow S$.*
- *A* **unary operation** *is a function that takes one element of S and outputs one element of S, i.e., $f\colon S \longrightarrow S$. An example of a unary operation is the inverse operation that takes x and returns x^{-1}.*

More types of operations:

Definition

- *A* **ternary operations** $f\colon S \times S \times S \longrightarrow S$
- *A n*-**ary operations** $f\colon \underbrace{S \times S \times \cdots \times S}_{n\ times} \longrightarrow S$
- *If* $n = 0$, *then we write the* 0*-ary product as the set with one element* $\{*\}$ *and a* 0**-ary operation** *is written as* $f\colon \{*\} \longrightarrow S$ *which basically picks out an element of* S*. Such an operation describes an element that does not change, i.e., a constant.*

Groups

Let us put this all together and give the formal definition of a group.

Definition

A **group** $(G, \star, e, (\ )^{-1})$ *is a set G with the following three operations:*

- *A binary operation: a function* $\star : G \times G \longrightarrow G$.
- *An identity: there is a special element e in G called the identity of the group. This can be stated in a functional way: there is a 0-ary operation* $u : \{*\} \longrightarrow G$ *where* $u(*) = e$.
- *An inverse operation: a unary operation* $(\ )^{-1} : G \longrightarrow G$

Groups

Definition

These three operations satisfy the following axioms:

- *The binary operation is associative: for all x, y and z, we have $(x \star y) \star z = x \star (y \star z)$.*
- *The identity acts like a unit of the binary operation (like when you multiply a number with 1, the result does not change, i.e., $1 \cdot n = n$ hence 1 is a "unit"): for all x, $x \star e = x = e \star x$.*
- *Applying the binary operation to an element with its inverse gives the identity: for all x in G, $x \star x^{-1} = e = x^{-1} \star x$.*

Groups

Example

- *The additive integers:* $(\mathbf{Z}, +, 0, -(\ ))$. *Addition and negation are the usual operations.*
- *The additive real numbers:* $(\mathbf{R}, +, 0, -(\ ))$. *Addition and negation are the usual operations.*
- *The multiplicative positive reals:* $(\mathbf{R}^+, \cdot, 1, (\ )^{-1})$ *where* $\mathbf{R}^+$ *are the positive real numbers, the operation* $\cdot$ *is multiplication, and the function* $(\ )^{-1}$ *takes* r *to* $\frac{1}{r}$.
- *Clock arithmetic:* $(\{0, 1, 2, 3, \ldots, 11\}, +, 0, -)$ *where addition and subtraction is going around the clock. 0 is the unit because when you add 0 to any number you get back to the original number. Notice that any non-negative integer would have worked.*
- *The* **trivial group**: $(\{0\}, +, 0, -)$. *This is the world's smallest group.*

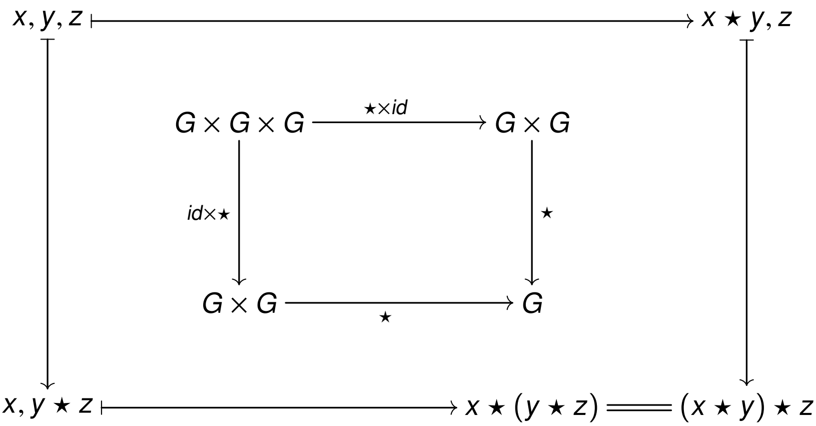
Central Idea

Important Categorical Idea

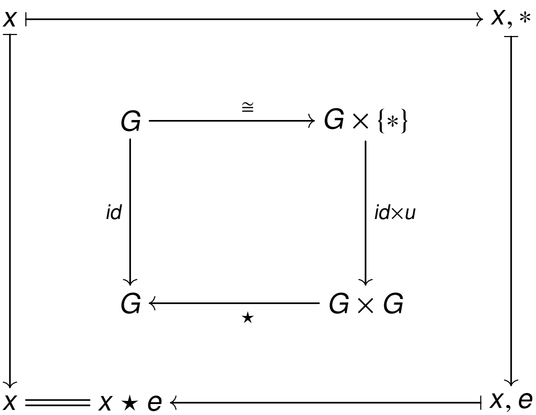
Descriptions Using Morphisms.

- *Many times, even when we have a nice, clear definition or description of a mathematical structure in terms of elements, we still desire a description in terms of functions or morphisms.*
- *The reason that a description using functions or morphisms is important is that once we have it, we can use it in many different categories.*
- *Whereas a description in terms of elements is good only in one context, a description in terms of functions or morphisms can be used in many different categories and contexts.*

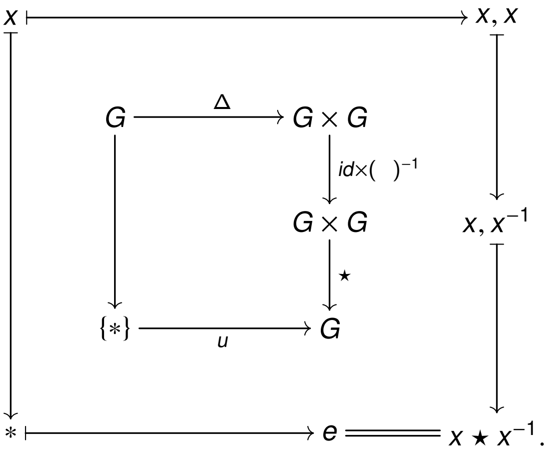
Parts of the three axioms of a group can be seen as commutative diagrams.



Groups



Groups



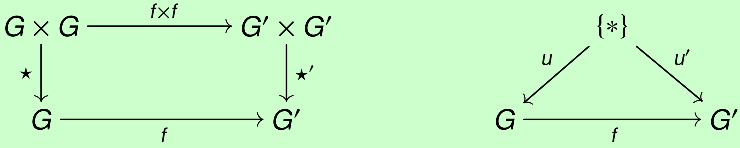
Groups

Just as a function is a way of mapping one set to another, and a graph homomorphism is a way of mapping one graph to another, a group homomorphism is a way of mapping one group to another.

Definition

Let $(G, \star, e, (\ \ )^{-1})$ and $(G', \star', e', (\ \ )'^{-1})$ be groups. A **group homomorphism** $f\colon (G, \star, e, (\ \ )^{-1}) \longrightarrow (G', \star', e', (\ \ )'^{-1})$ *is a function* $f\colon G \longrightarrow G'$ *that satisfies the following two axioms*

- *The function respects the group operation: for all $x, y \in G$, $f(x \star y) = f(x) \star' f(y)$*
- *The function respects the unit: $f(e) = e'$. Or:*



Groups

Example

Here are some examples of group homomorphisms.

- *There is always a unique group homomorphism from any group to the trivial group where every element of the group goes to* 0 *of the trivial group.*
- *There is always a unique group homomorphism from the trivial group to any group in which the* 0 *of the trivial group goes to the identity of the group.*
- *There is an inclusion group homomorphism* $inc: \mathbf{Z} \longrightarrow \mathbf{R}$.
- *There is a group homomorphism* $\mathbf{Z} \longrightarrow \{0, 1, 2, 3, \ldots, 11\}$ *that takes every whole number* x *and sends it to the remainder when* x *is divided by 12.*

Example

Here are some examples of group homomorphisms.

- *Let b be some positive real number called the "base". There is a exponential function* $b^{(\ )}\colon (\mathbf{R}, +, 0, -) \longrightarrow (\mathbf{R}^+, \cdot, 1, (\ )^{-1})$ *that takes a real number r and sends it to* b^r*. The two requirements to be a group homomorphism turn out to mean that* $b^{r+r'} = b^r \cdot b^{r'}$ *(*$b^{(\ )}$ *takes addition to multiplication) and* $b^0 = 1$*.*
- *There is a logarithm function (that is the inverse of the exponential function)* $Log_b\colon (\mathbf{R}^+, \cdot, 1, (\ )^{-1}) \longrightarrow (\mathbf{R}, +, 0, -)$*. The function* Log_b *takes a positive real number r to* $Log_b(r)$*. The requirements to be a group homomorphism are the well-known facts that* $Log_b(r \cdot r') = Log_b(r) + Log_b(r')$ *(*Log_b *takes multiplication to addition) and* $Log_b(1) = 0$*.*

Theorem

The composite of group homomorphisms is a group homomorphism. The composition is an associative operation.

Theorem

There is an **identity group homomorphism**, I_G, *for any group* G. *If* $f\colon G \longrightarrow G'$ *is a group homomorphism then* $f \circ I_G = f$ *and* $I_{G'} \circ f = f$.

Summary

- We met sets, and functons between sets.
- We saw that functions from and to a set are helpful in determining properties of the set.
- We discussed three operations on functions: composition, extension, and lifting.
- We met equivalence relations and quotient sets.
- Sets, graphs, and, groups have nice notions of maps between them. There are set functions, graph homomorphisms, and group homomorphisms. These maps have nice properties.

Foreshadow

- In the next Chapter we will meet the definition of a category.
- We will see that the collection of sets and functions between sets form a category.
- Similarly, we will see that the collection of graphs and graph homomorphisms between graphs form a category.
- And similarly, we will see that the collection of groups and group homomorphisms between groups form a category.
- In the next chapter we will see that many many other structures and maps between the structures form a category.