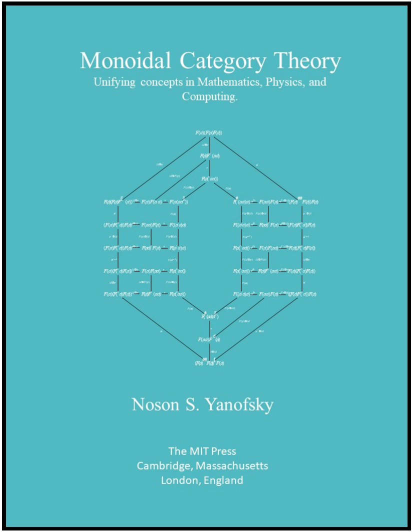
Monoidal Category Theory: Unifying concepts in Mathematics, Physics, and Computing



Chapter 2: Categories

Foreshadowing

- Chapter 2: Categories
  - Section 2.1: Basic Definitions and Examples
  - Section 2.2: Basic Properties
  - Section 2.3: Related Categories
  - Section 2.4: Mini-course: Basic Linear Algebra

Foreshadowing

- Chapter 2: Categories
  - Section 2.1: Definitions and Examples
    - Definition of a Category
    - Finite Categories
    - Examples from Computers
    - Examples from Logic
    - Examples from Mathematics
    - Examples from Physics
    - Examples from Computers
    - Another Definition of a Category

Foreshadowing

A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.

Paul Halmos

Remember

Before formally defining a category, let us summarize what we saw last Chapter concerning sets and functions. The collection of sets and functions form a category. By carefully examining this collection, we will see what is needed in the definition of a category.

Example

- *Consider the collection of all sets.*
- *There are functions between sets.*
- *If f is a function from set S to set T, then we write it as $f\colon S \longrightarrow T$. We call S the* **domain** *of f and T the* **codomain** *of f.*
- *Certain functions can be composed: for $f\colon S \longrightarrow T$ and $g\colon T \longrightarrow U$, there exists a function $g \circ f\colon S \longrightarrow U$ which is defined for s in S as $(g \circ f)(s) = g(f(s))$.*

Example

- *This composition operation is associative, which means that for $f: S \longrightarrow T$, $g: T \longrightarrow U$, and $h: U \longrightarrow V$, both ways of associating the functions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal to the function described as follows*

$$s \mapsto f(s) \mapsto g(f(s)) \mapsto h(g(f(s))).$$

- *That is, $h \circ (g \circ f) = (h \circ g) \circ f$ and on s of S this function has the value $h(g(f(s)))$.*
- *For every set S, there is a function $id_S : S \longrightarrow S$, which is called the identity function and is defined for s in S as $id_S(s) = s$.*
- *These identity functions have the following properties: for all $f: S \longrightarrow T$, it is true that $f \circ id_S = f$ and $id_T \circ f = f$.*

Remember

Example

- *The collection of sets and functions form a category called* $\mathbb{Set}$.
- *This category is easy to understand, and we use it to hone our ideas about many structures of category theory.*

Now for the formal definition of a category.

Definition

- *A* **category** $\mathbb{A}$ *is a collection of* **objects** $Ob(\mathbb{A})$ *and a collection of* **morphisms** $Mor(\mathbb{A})$ *which has the following structure:*
  - *Every morphism has an object associated to it called its domain: there is a function* $dom_{\mathbb{A}} : Mor(\mathbb{A}) \longrightarrow Ob(\mathbb{A})$.
  - *Every morphism has an object associated to it called its codomain: there is a function* $cod_{\mathbb{A}} : Mor(\mathbb{A}) \longrightarrow Ob(\mathbb{A})$.
  - *We write*

$$f\colon a \longrightarrow b \qquad \text{or} \qquad a \xrightarrow{\quad f \quad} b$$

for the fact that $dom_{\mathbb{A}}(f) = a$ *and* $cod_{\mathbb{A}}(f) = b$.

Definition (Continued.)

- *Adjoining morphisms can be composed: if* $f\colon a \longrightarrow b$ *and* $g\colon b \longrightarrow c$, *then there is an associated morphism* $g \circ f : a \longrightarrow c$. *We can write these morphisms as*

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & b \arrow[r, "g"] & c. \end{tikzcd}$$

- *Every object has an identity morphism: there is a function* $ident_{\mathbb{A}} : Ob(\mathbb{A}) \longrightarrow Mor(\mathbb{A})$. *We denote the identity of* a *as* $id_a : a \longrightarrow a$ *or*

$$\begin{tikzcd} a. \arrow[loop above, "id_a"] \end{tikzcd}$$

Definition (Continued.)

This structure must satisfy the following two axioms:

- *Composition is associative: given* $f\colon a \longrightarrow b$, $g\colon b \longrightarrow c$, *and* $h\colon c \longrightarrow d$, *the two ways of composing these maps are equal:*

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

  i.e., they are the same map from a *to* d.
- *Composition with the identity does not change the morphism: for any* $f\colon a \longrightarrow b$ *the composition with* id_a *is* f, *i.e.,* $f \circ id_a = f$, *and composition with* id_b *is also* f, *i.e.,* $id_b \circ f = f$.

Basic Examples

Example

Let us mention three examples of categories that we already saw in the last Chapter. Although we did not call them categories, the text and exercises showed that they each have the structure of a category.

- *Sets and functions form the category* $\mathbb{Set}$.
- *Directed graphs and graph homomorphisms give us* $\mathbb{Graph}$.
- *Groups and group homomorphisms make up* $\mathbb{Group}$.

Basic Definitions

The definition of a category is a "mouthful" that has many parts to it. There are several important comments concerning this definition.

Remark

- *We called the elements of Mor$(\mathbb{A})$ the "morphisms" of the category. We will also interchangeably use the words* **maps** *and* **arrows**.
- *It is important to notice that, if we have morphisms $f\colon a \longrightarrow b$ and $g\colon b \longrightarrow c$, then we write the composition as $g \circ f$ rather than $f \circ g$. We do this because in many categories the morphisms will be types of functions. When we apply the composition of functions, it looks like $g(f(\ \ ))$ which is notationally closer to $g \circ f$ than $f \circ g$. As we get more and more used to the language we will write gf rather than $g \circ f$.*

Remark (Continued.)

- *In the previous chapter we saw that for sets S and T we can look at the collection of all set functions from S to T.*
- *Now we look at the set of all morphisms between two objects. For objects a and b in category $\mathbb{A}$, there is a collection of all the morphisms from a to b which we write $Hom_{\mathbb{A}}(a, b)$. We call these collections* **Hom sets**.
- *Composition in the category in terms of the Hom sets becomes the operation*

$$\circ\colon Hom_{\mathbb{A}}(b, c) \times Hom_{\mathbb{A}}(a, b) \longrightarrow Hom_{\mathbb{A}}(a, c)$$

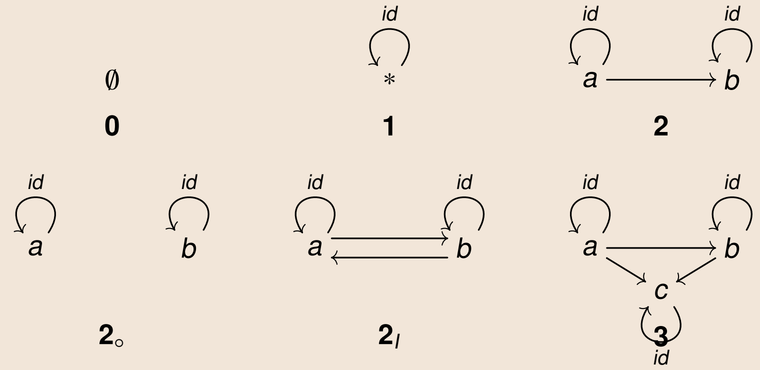
$$(g, f) \longmapsto g \circ f.$$

- *The fact that every element a of $\mathbb{A}$ has an identity element means that there is a special morphism in $Hom_{\mathbb{A}}(a, a)$ that satisfies the properties stated.*

Basic Examples

The categories $\mathbb{Set}$, $\mathbb{Graph}$, and $\mathbb{Group}$ each have collections of objects and morphisms that are infinite. Consider some examples of **finite categories** with finite objects and morphisms.

Example



Basic Examples

Example

- **0**, *the empty category, has no objects and no morphisms.*
- **1** *has 1 object and the single identity morphism on that object.*
- **2** *has 2 objects and 3 morphisms.*
- $\mathbf{2}_\circ$ *has the 2 objects and the 2 identity maps but does not have the non-identity morphism.*
- $\mathbf{2}_I$ *is like* **2** *but there are two non-identity morphisms, and their compositions are the identity morphisms. In total, it has 2 objects and 4 morphisms.*
- **3** *has 3 objects, 3 identity morphisms, and 3 non-identity morphisms.*

Although these categories may seem trivial, they will be very useful. They provide easy examples to explore concepts and have important roles as we explore category theory.

Basic Examples

Example

- *Not only does the collection of all sets form a category, but each individual set has the structure of a category.*
- *Let S be a set. We form the category $d(S)$ where the objects are the elements of S, and the only morphisms are identity morphisms.*
- *We call a category with only identity morphisms a* **discrete category**.
- *For example the set $S = \{a, b, c, d\}$ becomes the category:*

$$
\begin{tikzcd}
a \arrow[loop above, "id_a"] & b \arrow[loop above, "id_b"] \\
c \arrow[loop above, "id_c"] & d. \arrow[loop above, "id_d"]
\end{tikzcd}
$$

Examples from Computers

Example

- *The category of computable functions* $\mathbb{CompFunc}$ *is central for computer science.*
- *A function is* **computable** *if there exists a computer program that can tell a computer how to execute the function.*
- *That means, there is a computer program (written in some programming language) and if* $f(x) = y$ *then when* x *is entered into the computer as input, the program will output* y*.*
- *Computable functions have certain forms of data as input and output. The kind of data is called a* **type**. *Computers deal with types like* Nat *(natural numbers),* Int *(integers),* $Real$*,* $Bool$ *(Boolean),* $String$*, etc. The objects of* $\mathbb{CompFunc}$ *are sequences (or products) of types.*

Examples from Computers

Example (Continued.)

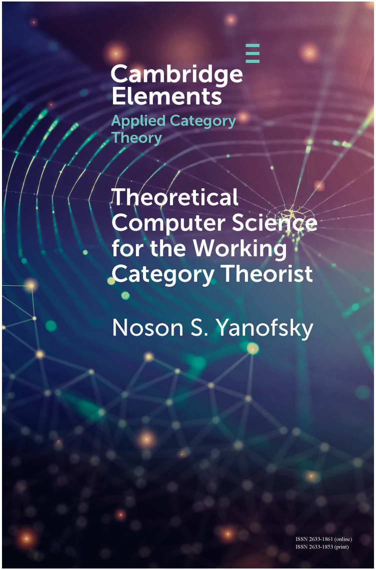
- *For example,* $Int \times Bool \times Bool \times Real$.
- *Given two sequences of types, a morphism of this category will be a computable function from the first sequence of types to the second sequence of types.*
- *A typical computable function might look like* $f\colon\ Int \times String \times Bool \longrightarrow Bool \times Real \times Real \times Nat$.

Example (Continued.)

- *Composition of two computable functions is easily seen to be a computable function (the program for the first program can be "composed" or "tagged onto" the program for the second function to form a program for the composition function.) Just like functions, composition of computable functions are associative.*
- *For every list of types there exists a (useless) computable function that accepts data of the appropriate type and outputs the same data without changing it. Such functions serve as the identity morphisms in this category. Composition with the identity functions does not change the function.*
- *(Notice that the name of this category comes from the morphisms, not the objects of the category.)*

Examples from Computers

This book is dedicated to the category of computable functions.



Examples from Logic

Example

- *The category* $\mathbb{Prop}$ *is about propositional logic.*
- *The objects of the category are propositional statements which are statements that are either true or false.*
- *Statements can be combined with logical operations like "and" (or "conjunction")* $\wedge$, *"or" (or "disjunction")* $\vee$, *"implication"* $\Rightarrow$, *"biconditional" (or "bi-implication")* $\Leftrightarrow$, *and "negation" (or "not")* $\neg$.
- *There is a single morphism from proposition* P *to proposition* Q *if and only if* P *logically implies (or entails)* Q.
- *For example, there are arrows* $P \wedge Q \longrightarrow P$, $P \wedge Q \longrightarrow Q$, *and* $P \longrightarrow P \vee Q$.

Examples from Logic

Example (Continued.)

- *The composition in the category exists because if P implies Q, i.e., $(P \longrightarrow Q)$ and Q implies R, i.e., $(Q \longrightarrow R)$, then it is obvious that P implies R, i.e., $(P \longrightarrow R)$.*
- *Associativity follows from the fact that there is at most one morphism between any two objects.*
- *The identities in the category come from the fact that for every propositional statement P, it is tautologically true that P implies P $(P \longrightarrow P)$.*
- *This category is different than the other infinite categories that we have already seen because between any two objects in the category there is either a single morphism or there is no morphism at all.*

Definition

- *A* **magma** $(M, \star)$ *is a set M with a binary operation (an operation with two inputs) $\star \colon M \times M \longrightarrow M$. This operation is called the "multiplication" of the magma. It is not assumed that this operation satisfies any axiom.*
- *A* **semigroup** $(M, \star)$ *is a magma whose binary operation is associative, i.e., for all x, y, and z in M, $x \star (y \star z) = (x \star y) \star z$.*
- *A* **monoid** $(M, \star, e)$ *is a semigroup whose binary operation has an identity element, that is, there exists an element e such that for all x in M, $x \star e = x = e \star x$.*

Examples from Mathematics

Magmas:

$(\mathbf{N}, +)$ $(\mathbf{Z}, +)$ $(\mathbf{Q}, +)$ $(\mathbf{R}, +)$ $(\mathbf{C}, +)$

$(\mathbf{N}, \cdot)$ $(\mathbf{Z}, \cdot)$ $(\mathbf{Q}, \cdot)$ $(\mathbf{R}, \cdot)$ $(\mathbf{C}, \cdot)$

Semigroups:

$(\mathbf{N}, +)$ $(\mathbf{Z}, +)$ $(\mathbf{Q}, +)$ $(\mathbf{R}, +)$ $(\mathbf{C}, +)$

$(\mathbf{N}, \cdot)$ $(\mathbf{Z}, \cdot)$ $(\mathbf{Q}, \cdot)$ $(\mathbf{R}, \cdot)$ $(\mathbf{C}, \cdot)$

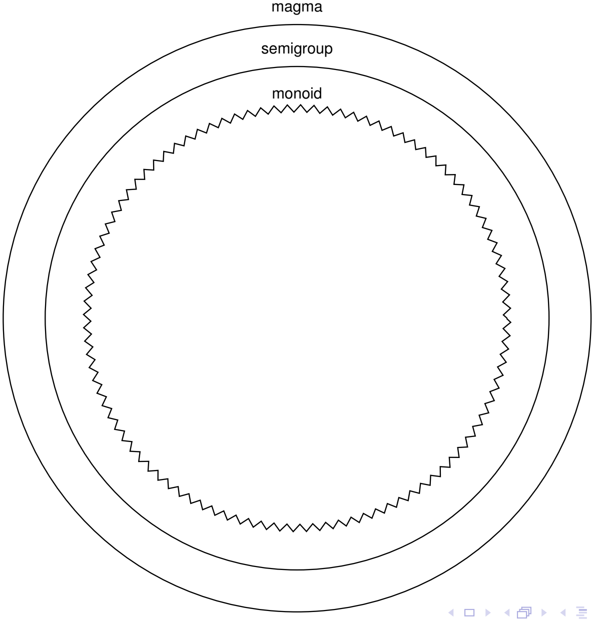
Monoids:

$(\mathbf{N}, +, 0)$ $(\mathbf{Z}, +, 0)$ $(\mathbf{Q}, +, 0)$ $(\mathbf{R}, +, 0)$ $(\mathbf{C}, +, 0)$

$(\mathbf{N}, \cdot, 1)$ $(\mathbf{Z}, \cdot, 1)$ $(\mathbf{Q}, \cdot, 1)$ $(\mathbf{R}, \cdot, 1)$ $(\mathbf{C}, \cdot, 1)$

Venn Diagram of Algebraic Structures

Venn Diagram of Algebraic Structures



Examples from Mathematics

Definition

- *A* **commutative monoid** $(M, \star, e)$ *is a monoid whose binary operation is commutative. That is, the multiplication satisfies the axiom that for all* x *and* y *in* M, $x \star y = y \star x$.
- *A* **group** $(M, \star, e, -(\ \ ))$ *is a monoid that has an inverse operation. That is, there is a function* $-(\ \ )\colon M \longrightarrow M$ *such that for all* x *in* M, $x \star -x = e = -x \star x$.
- *A* **commutative group** *or an* **abelian group** $(M, \star, e, -)$ *is a group whose binary operation is commutative. That is, the multiplication satisfies the axiom that for all* x *and* y *in* M, $x \star y = y \star x$. *Another way to think about it is as a commutative monoid with an inverse operation.*

Examples from Mathematics

Groups:

$$(\mathbf{Z}, +, 0, -) \quad (\mathbf{Q}, +, 0, -) \quad (\mathbf{R}, +, 0, -) \quad (\mathbf{C}, +, 0, -)$$

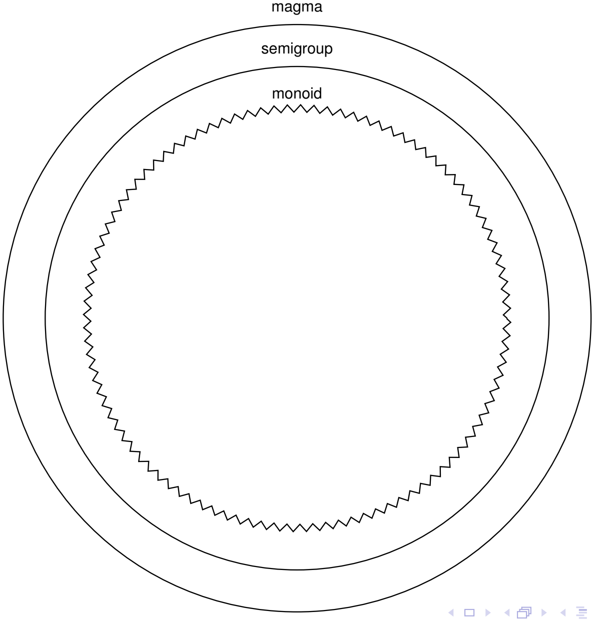
$$(\mathbf{Q}^+, \cdot, 1, (\ )^{-1}) \quad (\mathbf{R}^+, \cdot, 1, (\ )^{-1})$$

Abelian group:

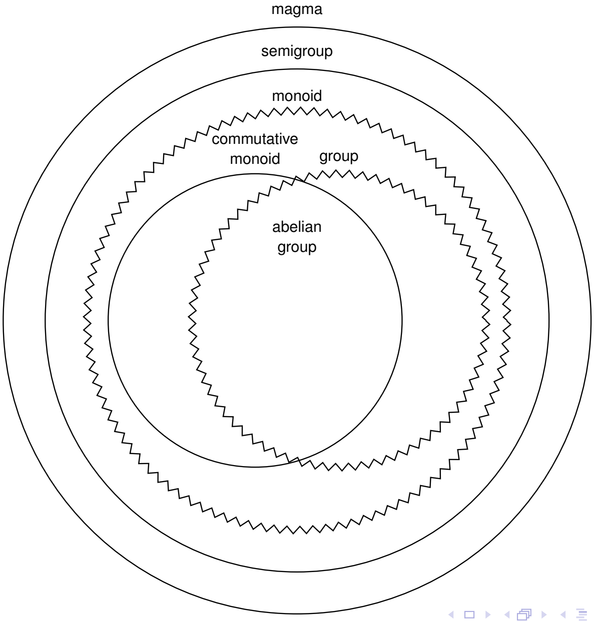
$$(\mathbf{Z}, +, 0, -) \quad (\mathbf{Q}, +, 0, -) \quad (\mathbf{R}, +, 0, -) \quad (\mathbf{C}, +, 0, -)$$

$$(\mathbf{Q}^+, \cdot, 1, (\ )^{-1}) \quad (\mathbf{R}^+, \cdot, 1, (\ )^{-1})$$

Venn Diagram of Algebraic Structures



Venn Diagram of Algebraic Structures



Definition

- *A* **ring** $(M, \star, e, -, \odot, u)$ *is an abelian group with another binary associative operation* $\odot\colon M \times M \longrightarrow M$ *and another identity element* u *(i.e. for all* x *in* M*, we have* $x \odot u = x = u \odot x$*, or another way to say that is* $(M, \odot, u)$ *forms a monoid) and for which the new operation distributes over the old operation. That means that for all* x*,* y*, and* z *in* M

$$x\odot(y\star z) = (x\odot y)\star(x\odot z) \quad \textit{and} \quad (y\star z)\odot x = (y\odot x)\star(z\odot x).$$

- *A* **field** $(M, \star, e, -, \odot, u, (\ \ )^{-1})$ *is a ring with a partial inverse for the second binary operation. This means that there is an operation* $(\ \ )^{-1} : M \longrightarrow M$ *which is defined for all* x *in* M *except the identity element* e*. The inverse operation satisfies the axiom: for all* $x \neq e, x \odot x^{-1} = u = x^{-1} \odot x$.

Examples from Mathematics

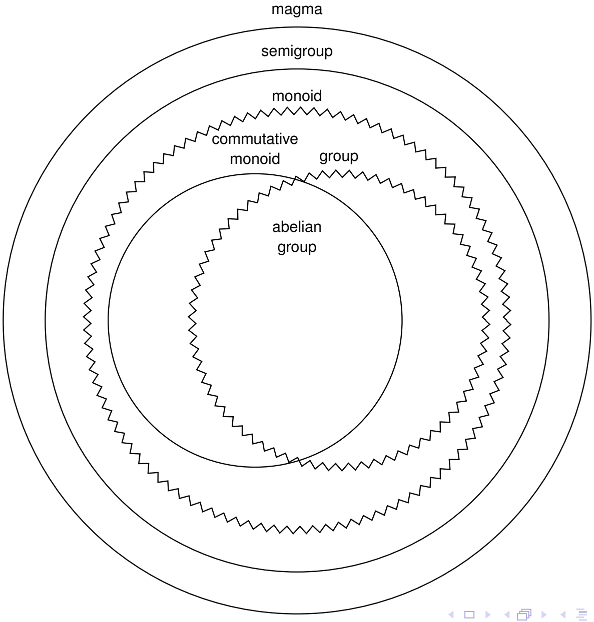
Rings:

$$(\mathbf{Z}, +, 0, -, \cdot, 1) \quad (\mathbf{Q}, +, 0, -, \cdot, 1) \quad (\mathbf{R}, +, 0, -, \cdot, 1) \quad (\mathbf{C}, +, 0, -, \cdot, 1)$$

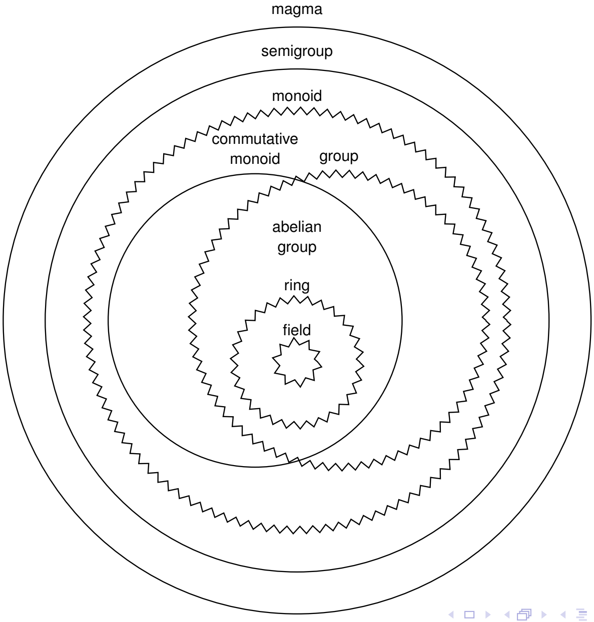
Fields:

$$(\mathbf{Q}, +, 0, -, \cdot, 1, ()^{-1}) \quad (\mathbf{R}, +, 0, -, \cdot, 1, ()^{-1}) \quad (\mathbf{C}, +, 0, -, \cdot, 1, ()^{-1}).$$

Venn Diagram of Algebraic Structures



Venn Diagram of Algebraic Structures



Examples from Mathematics

Monoids will play a major role in this text so it pays to spell out all the details of their definition and to state them with commutative diagrams.

Definition

A **monoid** *is a triple* $(M, \star, e)$ *where*

- *M is a set of elements,*
- $\star : M \times M \longrightarrow M$ *is a set function, i.e., a binary operation, and*
- *e is an element of M, i.e., there is a set function that picks out e in M,* $\upsilon : \{*\} \longrightarrow M$.

These ingredients must satisfy the following requirements: $\star$ *is associative, and e must behave like a unit, i.e. the commutativity of the following two diagrams*

Definition (Continued.)

$$
\begin{tikzcd}
M \times M \times M \arrow[r, "id_M \times \star"] \arrow[d, "\star \times id_M"'] & M \times M \arrow[d, "\star"] \\
M \times M \arrow[r, "\star"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\{*\} \times M \arrow[r, "\upsilon \times id_M"] \arrow[dr, "\cong"'] & M \times M \arrow[d, "\star"] & M \times \{*\} \arrow[l, "id_M \times \upsilon"'] \arrow[dl, "\cong"] \\
& M. &
\end{tikzcd}
$$

Examples from Mathematics

An example of a monoid that will arise over and over again is the following.

Example

- *Let $\mathbb{A}$ be any category and a be any object in $\mathbb{A}$, then consider all the morphisms that start and end at a, i.e. $Hom_{\mathbb{A}}(a, a)$.*
- *A morphism that starts and ends at the same object is called an* **endomorphism**.
- *We write this collection as $End(a)$ and it forms a monoid.*
- *Given $f\colon a \longrightarrow a$ and $g\colon a \longrightarrow a$, we can multiply them as $f \circ g$.*
- *The multiplication is associative because the composition in $\mathbb{A}$ is associative. The unit is the identity morphism id_a because for all f, we have $f \circ id_a = f = id_a \circ f$.*

Examples from Mathematics

- Following an earlier Important Categorical Idea, we must discuss morphisms between each of these algebraic structures.
- For each type of algebraic structure there is a notion of a **homomorphism** from one structure to another structure of the same type.
- If M and M' are of the same type of structure, then a homomorphism $f\colon M \longrightarrow M'$ is a set function from M to M' that "respects" or "preserves" all the operations.

Examples from Mathematics

- For example, if $(M, \star, e, -, \odot, u)$ and $(M', \star', e', -', \odot', u')$ are both rings, then a homomorphism of rings is a set function $f\colon M \longrightarrow M'$ that satisfies the following axioms:
  - f must respect the multiplication operations: for all x, y in M, $f(x \star y) = f(x) \star' f(y)$.
  - f must respect the identity elements: $f(e) = e'$.
  - f must respect the inverse operations: for all x in M, $f(-x) = -'(f(x))$.
  - f must respect the other binary operations: for all x, y in M, $f(x \odot y) = f(x) \odot' f(y)$.
  - f must respect the other identity elements: $f(u) = u'$.

Because of the importance of monoids in this book, let us work out the details of being a monoid homomorphism in terms of diagrams.

Definition

Let $(M, \star, \upsilon)$ and $(M', \star', \upsilon')$ be monoids and $f\colon M \longrightarrow M'$ be a set function, then f is a **monoid homomorphism** *if it respects the multiplications and the units. This means these two diagrams commute:*

$$\begin{tikzcd}
M \times M \arrow[r, "f\times f"] \arrow[d, "\star"'] & M' \times M' \arrow[d, "\star'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
& \{*\} \arrow[dl, "\upsilon"'] \arrow[dr, "\upsilon'"] & \\
M \arrow[rr, "f"'] & & M'.
\end{tikzcd}$$

Examples from Mathematics

Theorem

The composition of homomorphisms is a homomorphism and the composition operation is associative.

Proof.

Let M, M', and M'' be some type of structure with binary operations $\star, \star', \star''$, respectively. If $f\colon M \longrightarrow M'$ and $g\colon M' \longrightarrow M''$ are homomorphisms, then we have

$$\begin{aligned}
(g \circ f)(x \star y) &= g(f(x \star y)) && \text{by def of composition}\\
&= g(f(x) \star' f(y)) && \text{bec } f \text{ is a homomorphism}\\
&= g(f(x)) \star'' g(f(y)) && \text{bec } g \text{ is a homomorphism}\\
&= (g \circ f)(x) \star'' (g \circ f)(y) && \text{by def of composition}
\end{aligned}$$

Hence $g \circ f\colon M \longrightarrow M''$ is a homomorphism. This composition is associative because they are basically set functions. $\square$

Examples from Mathematics

Each of the algebraic structures defined here (and much more) together with their homomorphisms give an example of a category.

Example

For these algebraic structures, the collection of all the structures and their homomorphisms form a category. This gives us the categories $\mathbb{Magma}$, $\mathbb{SemiGp}$, $\mathbb{Monoid}$, $\mathbb{ComMonoid}$, $\mathbb{Group}$, $\mathbb{AbGp}$, $\mathbb{Ring}$ *and* $\mathbb{Field}$. *The relationships between these categories will be examined in Chapter 4.*

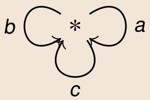
Example

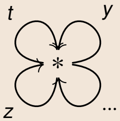
- *Not only does the entire collection of monoids form a category, but each individual monoid can be seen as a special type of category.*
- *If* $(M, \star, e)$ *is a monoid, then there exists a category* $A(M, \star, e)$ *or just* $A(M)$ *whose morphisms are the elements of the monoid.*
- *The category consists of a single object* $*$, *and the morphisms in* $A(M)$ *are the elements of* M *and they come and go from* $*$.

Examples from Mathematics

Example (Continued)

- *Such a category is called a* **one-object category** *or a* **single-object category**. *We might visualize this as in the following the two examples:*





- *Composition of morphisms are given by the monoid multiplication. That is, if there are elements of the monoid* $m\colon * \longrightarrow *$ *and* $m'\colon * \longrightarrow *$, *then their composition is* $m' \star m\colon * \longrightarrow *$. *The identity element of the monoid,* e, *becomes the identity morphism* id_* *in the category* $A(M)$.

Examples from Mathematics

We close our list of mathematical examples of categories with the notion of a topological space. This structure is one of the most important structures in modern mathematics and physics.

Definition

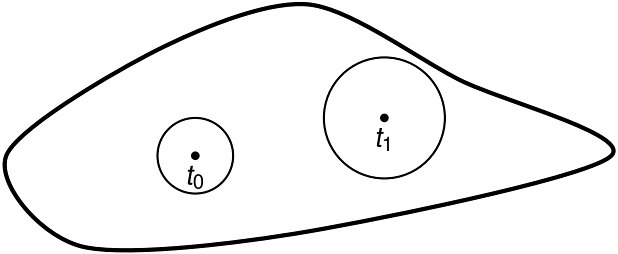
A **topological space** (T, τ) *is a set* T *(the elements here are called "points") and* τ *a set of subsets of* T *that are called* **open sets**. *The subsets* τ *satisfies the following requirements:*

- *the empty set* $\emptyset$ *and the entire set* T *are in* τ*,*
- *the set* τ *is closed under finite intersection: given a finite set of subsets in* τ*, their intersection is in* τ*, and*
- *the set* τ *is closed under arbitrary union: given any set of subsets in* τ*, their union is in* τ*.*

The collection τ *is called the* **topology of** T.

Examples from Mathematics

The intuition is that the open sets determine which elements can be distinguished by maps in the category. If t_0 and t_1 are in the same open set, then it is going to be hard to distinguish them. For typical spaces, an open set around a point contains all the points near it. Here is a typical space with points t_0 and t_1 and open sets around those points.



Examples from Mathematics

Example

- *For every set of points T, there are two extreme examples of topologies on T.*
- *If τ is the set of all subsets of T, i.e., the power set of T, then T is said to have the* **discrete topology**.
- *If τ only consists of the empty set $\emptyset$ and the entire set T, then all three requirements are satisfied and T is said to have the* **indiscrete topology** *or the* **trivial topology**.
- *The discrete topology has the most open sets possible, and the indiscrete topology has the least open sets possible. The other topologies fall somewhere between these two extremes.*

Examples from Mathematics

The definition of a topological space goes hand-in-hand with the definition of a map between topological spaces. Such maps are called “continuous maps.”

Definition

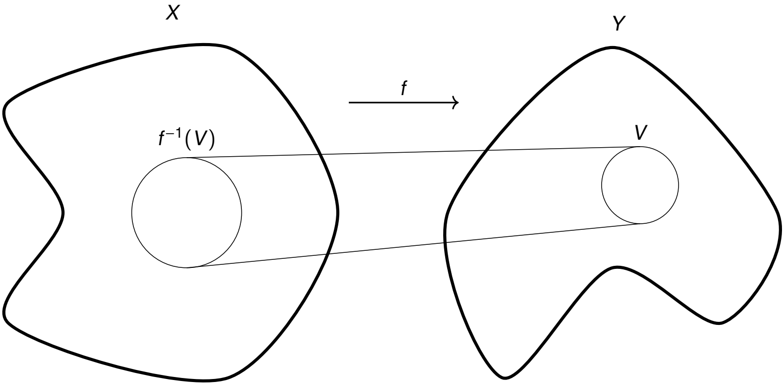
Given topological spaces (X,τ) and (Y,σ), a **continuous map** *f from (X,τ) to (Y,σ), written $f\colon (X,\tau) \longrightarrow (Y,\sigma)$ is a set function $f\colon X \longrightarrow Y$ that satisfies the following requirement:*

$$\text{for every open set } V \in \sigma, \text{the preimage } f^{-1}(V) =$$

$$\{x \in X : f(x) \in V\} \subset X \text{ is an open set in } \tau.$$

Notice that the requirement, in a sense, goes “backwards.” We do not require open sets to go to open sets. Rather, we require that open sets come from open sets.

Examples from Mathematics



A continuous map. The preimage of an open set is an open set.

Examples from Mathematics

The following is a theorem about the extreme topologies.

Theorem

Consider topological spaces (X, τ) and (Y, σ), if

- *σ is the indiscrete topology, or*
- *τ is the discrete topology*

then any function $f\colon X \longrightarrow Y$ is a continuous map.

Proof.

- If σ is the indiscrete topology, its only two open sets are the $\emptyset$ and Y. Then $f^{-1}(\emptyset) = \emptyset \subset X$ and, since f takes every element of X to some element in Y, we have $f^{-1}(Y) = X$, which are both in τ.
- If τ is the discrete topology, then no matter what open set is in σ, the preimage of it is in τ because τ has every subset of X.

$\square$

Examples from Mathematics

Exercise

Show that the composition of two continuous maps is a continuous map.

Exercise

Show that the composition operation is associative.

Exercise

Show that the identity function $id_T : (T, \tau) \longrightarrow (T, \tau)$ *is a continuous map and it acts like a unit to the composition operation.*

Summing up these Exercises gives us the following category of topological spaces.

Example

The collection of all topological spaces and continuous maps form the category $\mathbb{T}\mathbb{op}$.

Examples from Physics

Our physics examples are mostly (mathematical) structures that are used by physicists to describe physical phenomena. We will discuss vector spaces, manifolds, and matrices.

- Physicists use vector spaces to describe directions and lengths.
- They use manifolds to describe phenomena with enough structure to perform calculus-like operations.
- Matrices are arrays of numbers that store multidimensional information.

Examples from Physics

We will define and work with vector spaces the mini-course on linear algebra. Suffice it to say the following.

Example

For every field $\mathbf{K}$*, the category of* $\mathbf{K}$*-vector spaces and linear transformations form a category* $\mathbf{K}\mathbb{V}\mathbb{ect}$*. We will mostly be concerned with* $\mathbf{R}\mathbb{V}\mathbb{ect}$ *and* $\mathbf{C}\mathbb{V}\mathbb{ect}$*.*

Examples from Physics

Physicists (and mathematicians) work with manifolds. These are "nice" topological spaces.

Definition

A smooth n-dimensional **manifold** *or a n-***manifold** *is a topological space that has the property that at every point there is a surrounding open set that "looks like $\mathbf{R}^n$." We say it is "locally $\mathbf{R}^n$."*

Rather than getting into the nitty-gritty details of the definition, let us look at many examples.

Example

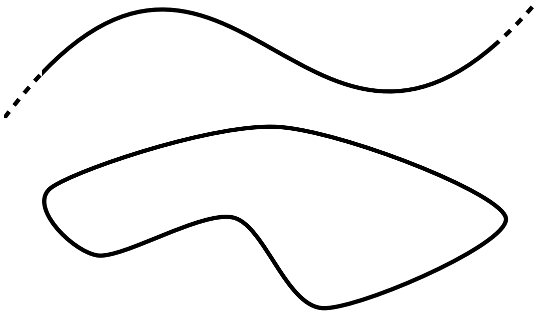
Consider the surface of planet Earth. We all know that the Earth is a sphere (with some flattening at the poles). However, when we look around us, Earth looks like a flat plane, i.e., it looks like $\mathbf{R}^2$. Therefore the surface of the Earth is a 2-dimensional manifold. For that matter, the surface of every sphere is a 2-manifold.

Example

Let us go through some examples.

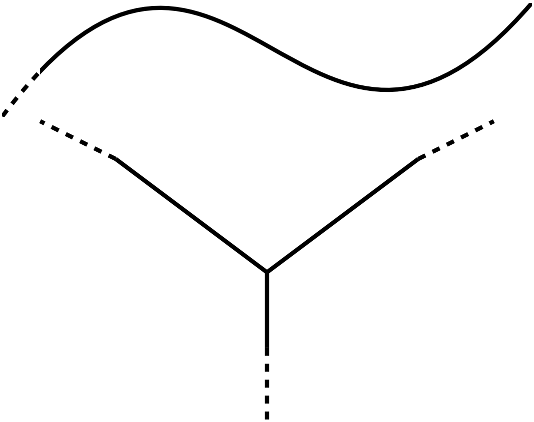
- *A point is a* 0*-dimensional manifold.*
- *The empty set is an n-manifold for all n because it is true that every point in the empty set is locally* $\mathbf{R}^n$*.*
- *There are essentially two types of* 1*-dimensional manifolds: an open line and something closed like a circle.*
- *It is important that the ends of the curved line be "open" (as in* $(0,1) \subseteq \mathbf{R}$ *as opposed to* $[0,1] \subseteq \mathbf{R}$ *which is "closed.") Any point on the line, regardless how close the point is to the end, has a small neighborhood that looks like an "open" part of the real line.*

Examples from Physics



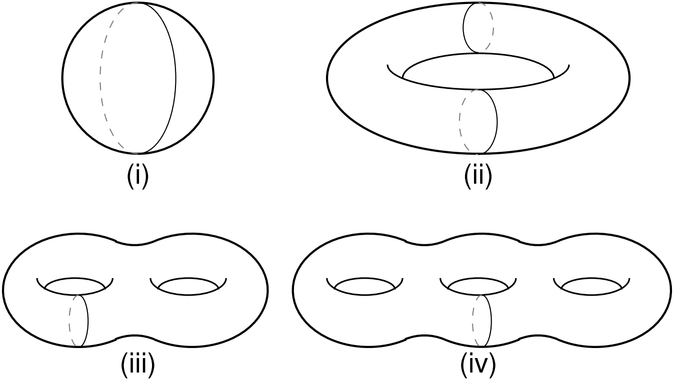
Two 1-dimensional manifolds: an open line and a closed like circle.

Examples from Physics



Two topological spaces that are not manifolds. The curved line on top is not a 1-manifold because the end point at the right is not locally like the real line. The “Y” on the right is not a manifold because the one point in the middle has three lines coming out of it, and every point in **R** has two lines coming out of it. At that

Examples from Physics



Four 2-manifolds. (i) is a sphere, (ii) is a doughnut with one hole, (iii) is a doughnut with two holes and (iv) is a doughnut with three holes. We can go on and discuss a doughnut with *n* holes. It is a theorem that these are all the finite (the technical term is **compact**) 2-manifolds. The number of holes is called the **genus** of the 2-manifold. The sphere has genus 0.

Examples from Physics

Let us talk about maps between manifolds.

Definition

A map between manifolds is called a **smooth map** *if it is a continuous map of topological spaces and it respects all the local* $\mathbf{R}^n$ *structure. (Again, we ignore the technical details of the definition, because we will not be using the details.)*

Example

The n-dimensional manifolds and smooth maps form a category n-$\mathbb{Manif}$*. All the different dimensional manifolds and all the possible smooth maps can be combined to form a category* $\mathbb{Manif}$*.*

Examples from Physics

Example

The set of all matrices with real number entries forms a category $\mathbf{R}\mathbb{M}\mathrm{at}$*. The objects are the natural numbers, and the set of all morphisms from m to n is the set of n by m matrices with real number entries. In order to avoid confusion, we will write a matrix* $A : m \longrightarrow n$ *as* A_{nm}*. The composition in* $\mathbf{R}\mathbb{M}\mathrm{at}$ *is simply matrix multiplication. In detail, if* A_{nm} *is a m by n matrix and* B_{pn} *is a n by p matrix, then the matrix multiplication* $B_{pn} \cdot A_{nm}$ *is the composition of the morphisms as depicted here:*

$$\begin{tikzcd} m \arrow[r, "A_{nm}"'] \arrow[rr, bend left, "B_{pn} \cdot A_{nm}"] & n \arrow[r, "B_{pn}"'] & p. \end{tikzcd}$$

For every n, we say that there is a unique "empty" matrix in $Hom_{\mathbf{R}\mathbb{M}\mathrm{at}}(0, n)$ *and similarly in* $Hom_{\mathbf{R}\mathbb{M}\mathrm{at}}(n, 0)$*. Like the elements in an empty set, such matrices do not exist. However, we need to*

Example

Consider the following example of multiplying matrices with entries in **N**.

$$\begin{tikzcd}
4 \arrow[r, "\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}"'] \arrow[rr, bend left, "\begin{bmatrix} 38 & 44 & 50 & 56 \\ 83 & 98 & 113 & 128 \end{bmatrix}"] & 3 \arrow[r, "\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}"'] & 2.
\end{tikzcd}$$

Examples from Computers and Logic

We met the category $\mathbb{Set}$ of sets and functions between sets. There is another category with sets as objects which contains more morphisms. Relations are generalizations of functions that describe connections between sets.

Definition

A **relation** *R from set S to set T, written $R\colon S \not\longrightarrow T$, is a subset of $S \times T$. That is, $R \subseteq S \times T$.*

Examples from Computers and Logic

A function is a special type of relation.

Example

Every function $f\colon S \longrightarrow T$ can be seen as a relation $\hat{f}\colon S \not\longrightarrow T$

$$\hat{f} = \{(s,t) : f(s) = t\} \subseteq S \times T.$$

Relations are more general than functions because (i) some elements in S might not be related to any element in T, and, furthermore, in a relation (ii) some elements in S might be related to more than one element in T.

Relations can be composed. Given a relation $R\colon S \not\longrightarrow T$ and $Q\colon T \not\longrightarrow U$, the composition is written as $Q \circ R\colon S \not\longrightarrow U$ and is defined as

$$Q \circ R = \{(s,u) : \text{there exists a } t \in T \text{ such that } (s,t) \in R \text{ and } (t,u) \in Q\}.$$

Examples from Computers and Logic

Example

Consider the following three sets $S = \{a, b, c, d, e\}$, $T = \{w, x, y, z\}$ *and* $U = \{1, 2, 3, 4\}$. *Let* $R\colon S \not\longrightarrow T$ *and* $Q\colon T \not\longrightarrow U$ *be defined as* $R = \{(a, x), (a, y), (c, x), (d, z)\}$ *and* $Q = \{(w, 4), (x, 2), (y, 2)\}$. *Then* $Q \circ R = \{(a, 2), (c, 2)\}$

We can form the category of sets and relations.

Example

The category $\mathbb{R}\mathrm{el}$ of relations has sets as objects and relations between sets as morphisms.

Definition

Some relations have special properties. Here are several possible properties of relations. The relation $R \colon S \not\longrightarrow T$ *is*

- **total-valued** *if for every* $s \in S$ *there is at least one element* $t \in T$ *such that* $(s, t) \in R$.
- **single-valued** *if for every* $s \in S$ *there is at most one* $t \in T$ *such that* $(s, t) \in R$. *In other words, if* $(s, t) \in R$ *and* $(s, t') \in R$ *then* $t = t'$.
- **one-to-one** *if for every* $t \in T$ *there is at most one* $s \in S$ *such that* $(s, t) \in R$. *In other words, if* $(s, t) \in R$ *and* $(s', t) \in R$ *then* $s = s'$.
- **onto** *if for every* $t \in T$ *there is an* $s \in S$ *such that* $(s, t) \in R$.

While, in general, a relation is from one set to another set, it is also possible to have a relation from a set to itself. Such a relation tells how elements in a set are related to themselves. There are also special properties of relations from a set to itself. The relation

Examples from Computers and Logic

Definition

For every relation $R\colon S \not\longrightarrow T$*, there is a related* **inverse relation** $R^{-1}\colon T \not\longrightarrow S$ *that is defined as*

$$R^{-1} = \{(t,s) : (s,t) \in R\} \subseteq T \times S.$$

Example

Let $S = \{a,b,c,d,e\}$ *and* $T = \{w,x,y,z\}$ *be sets and* $R\colon S \not\longrightarrow T$ *be defined as* $R = \{(a,x),(a,y),(c,x),(d,z)\}$*, then* $R^{-1}\colon T \not\longrightarrow S$ *is* $\{(x,a),(y,a),(x,c),(z,d)\}$*.*

We saw that properties of functions can be understood in terms of composition of functions, here, the properties of relations can be understood in terms of composition of relations.

Theorem

Relation $R\colon S \not\longrightarrow T$ has the property of being

- *total-valued if $id_S \subseteq R^{-1} \circ R$.*
- *single-valued if $R \circ R^{-1} \subseteq id_T$.*
- *one-to-one if $R^{-1} \circ R \subseteq id_S$.*
- *onto if $id_T \subseteq R \circ R^{-1}$.*

There are similar methods to describe properties of relations from a set to itself.

Theorem

Relation $R\colon S \not\longrightarrow S$ has the property of being

- *reflexive if $id_S \subseteq R$.*
- *symmetric if $R = R^{-1}$.*
- *transitive if $R \circ R \subseteq R$.*
- *anti-symmetric if $R \cap R^{-1} \subseteq id_S$.*
- *total if $R \cup R^{-1} = S \times S$.*

Proof.

We will prove that a relation is total-valued if and only if $id_S \subseteq R^{-1} \circ R$.

$$
\begin{aligned}
R \text{ total-valued} &\iff \text{for any } s \in S \text{ there is a } t \in T \text{ with } (s,t) \in R \\
&\iff \text{for any } s \in S \text{ there is a } t \in T \text{ with } (t,s) \in R^{-1} \\
&\iff \text{for any } s \in S \text{ there is a } t \in T \text{ with } (s,t) \in R \\
&\qquad \text{and } (t,s) \in R^{-1} \\
&\iff \text{for any } s \in S \text{ we have } (s,s) \in R^{-1} \circ R \\
&\iff id_S \subseteq R^{-1} \circ R.
\end{aligned}
$$

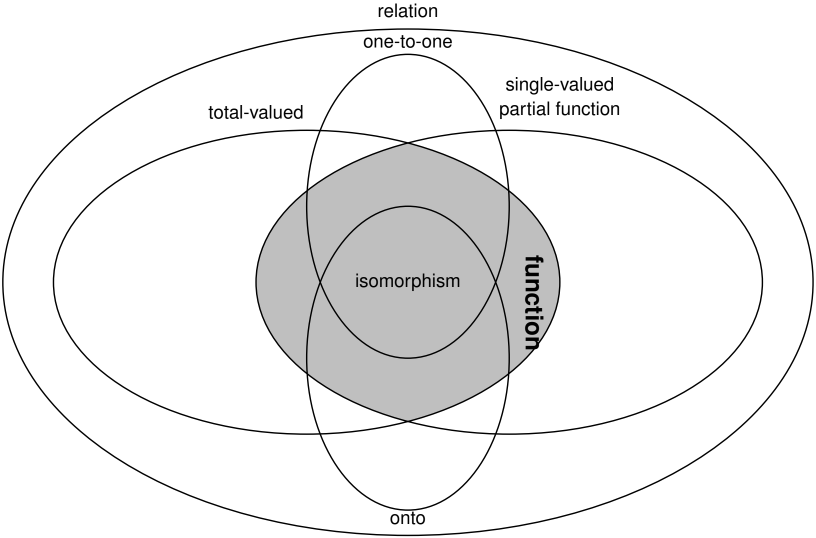
□

Definition

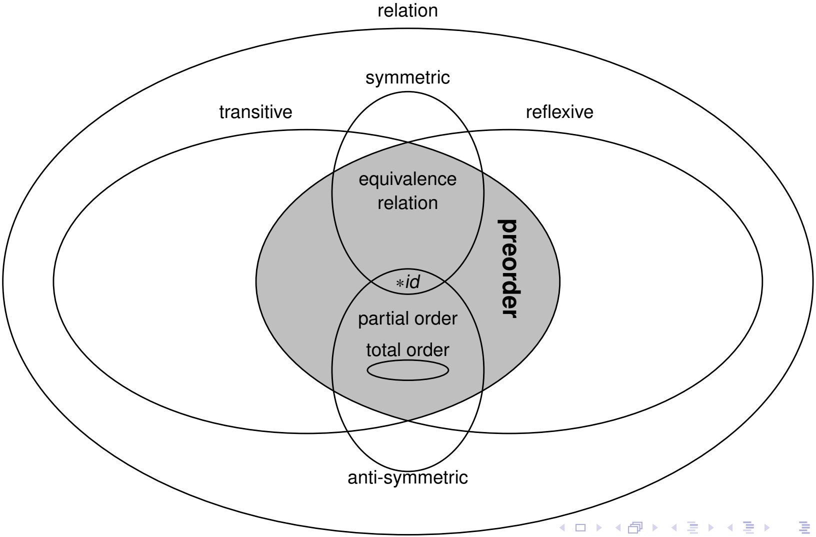
We can use these properties to construct many definitions about arbitrary relations between sets.

- *A* **partial function** *is a relation that is single-valued.*
- *A* **function** *is a partial function that is total-valued, i.e., a relation that is total-valued and single-valued.*
- *A* **one-to-one function** *is a function that is one-to-one.*
- *An* **onto function** *is a function that is onto.*
- *An* **isomorphism** *is a function that is one-to-one and onto.*

Venn Diagrams of Types of Relations



Venn Diagram of Types of Relations Between a Set and Itself



We can also make definitions about relations between a set and itself.

Definition

- *A* **preorder** *is a relation from a set to itself that is reflexive and transitive.*
- *A* **partial order** *is a preorder that is anti-symmetric, i.e., it is a relation that is reflexive, transitive, and anti-symmetric.*
- *A* **total order** *is a partial order that is total, i.e., it is a relation that is reflexive, transitive, anti-symmetric and total.*
- *An* **equivalence relation** *is a relation from a set to itself that is reflexive, symmetric, and transitive.*

These definitions are very important and need to be memorized. The notions of preorder and partial order will appear on almost every page of the rest of the book.

Partial functions compose like relations. In detail, if $f\colon S \longrightarrow T$ and $g\colon T \longrightarrow U$, then $g \circ f\colon S \longrightarrow U$ is a partial function and on input s, it will have a the value $g(f(s))$ if s is defined for f and g is defined for $f(s)$. The identity relation is also a partial function.

Example

The category $\mathbb{P}\mathrm{ar}$ of partial functions has sets as objects and partial functions between the sets as morphisms. Composition and identity functions are the same as in $\mathbb{R}\mathrm{el}$.

- In summary, the categories, $\mathbb{Rel}$, $\mathbb{Par}$, and $\mathbb{Set}$ all have sets as objects, all have the same composition, and all have the same identity morphisms.
- $\mathbb{Rel}$ is the biggest of the three and $\mathbb{Set}$ is the smallest of the three.
- (Since all three categories have sets as objects, it is disingenuous for $\mathbb{Set}$ to have its name. The category $\mathbb{Set}$ should really be called $\mathbb{Fun}$.)

Let us focus on partial orders and restate the definition.

Definition

A **partial order** $(P, \leq)$ *is a set P and a relation $\leq \subseteq P \times P$ that satisfies the following requirements:*

- *$\leq$ is reflexive: for all $p \in P$, $p \leq p$,*
- *$\leq$ is transitive: for all p, q and r in P, if $p \leq q$ and $q \leq r$ then $p \leq r$, and*
- *$\leq$ is antisymmetric: for all p and q in P, if $p \leq q$ and $q \leq p$ then $p = q$.*

(Our use of the symbol $\leq$ is not to be confused with the standard use of the same symbol with numbers.

As typical category theorists, right after defining a structure, we are interested in defining the morphisms between structures of this type.

Definition

Let $(P, \leq)$ and $(Q, \preceq)$ be two partial orders. An **order preserving function** $f: (P, \leq) \longrightarrow (Q, \preceq)$ *is a function from P to Q that satisfies the following axiom: for all p and p' in P, if $p \leq p'$ then $f(p) \preceq f(p')$.*

Exercise

Show that the composition of two order preserving functions is order preserving. Furthermore, show that the composition is associative.

Exercise

Show that the identity function is order preserving and that the identity order preserving function acts like a unit to the

Example

Not only does the collection of all partial orders form a category, but each individual partial order forms a category. Let $(P, \leq)$ be a partial order. Consider the category $B(P, \leq)$ or $B(P)$ whose objects are the elements of P and a single morphism $p \longrightarrow q$ if and only if $p \leq q$. The transitivity of $\leq$ assures us that the composition works. The fact that there is at most one morphism between any two objects shows us that the composition is associative. The reflexivity of $\leq$ corresponds to the identity maps. We sometimes abuse notation and write P for the category.

Example

Everything we said about partial orders is true about preorders. Hence there is a category of all preorders and order preserving maps which we call $\mathbb{P}\mathrm{re}\mathbb{O}$. Furthermore, every preorder in $(P, \leq)$ forms a category.

Central Idea

Important Categorical Idea

Contexts Are Central. *Certain structures have arisen several times in different contexts. Consider the rational numbers* $\mathbf{Q}$*, the real numbers* $\mathbf{R}$*, and the complex numbers* $\mathbf{C}$*. They are (to name a few)*

- *Sets of numbers and hence objects of* $\mathbb{Set}$*.*
- *Algebraic objects, i.e., objects of the categories* $\mathbb{Magma}$*,* $\mathbb{SemiGp}$*,* $\mathbb{Monoid}$*,* $\mathbb{Group}$*,* $\mathbb{AbGp}$*,* $\mathbb{Ring}$*, and* $\mathbb{Field}$*.* $\mathbf{R}$ *is also a real vector space, while* $\mathbf{C}$ *is a real vector space and a complex vector space.*
- *As monoids they each are also single object categories* $A(\mathbf{Q}), A(\mathbf{R})$*, and* $A(\mathbf{C})$*.*
- *They are topological objects, i.e., they can be given topologies and hence are objects in* $\mathbb{Top}$*.* $\mathbf{R}$ *is also a 1-dimensional manifold while* $\mathbf{C}$ *is a 2-dimensional real manifold.*

Important Categorical Idea (Continued.)

Contexts Are Central.

- *They are objects in the categories* $\mathbb{Par}$ *and* $\mathbb{Rel}$*.*
- *They are partial orders and hence objects in* $\mathbb{PO}$ *.*
- *Each one is a partial order category:* $B(\mathbf{Q})$*,* $B(\mathbf{R})$*, and* $B(\mathbf{C})$*.*

This fact that one concept can be seen in many different contexts is sometimes confusing to the beginner. However, with time, one gets the hang of it. This possible confusion forces us to specify the context of the concept that we are discussing.

Let us stress this point with an example. While a group theorist looks at the real numbers one way, a topologist looks at the real numbers in another way, and a physicist and a computer scientist look at the real numbers in their own ways, the category theorist must look at the real numbers in all possible ways.

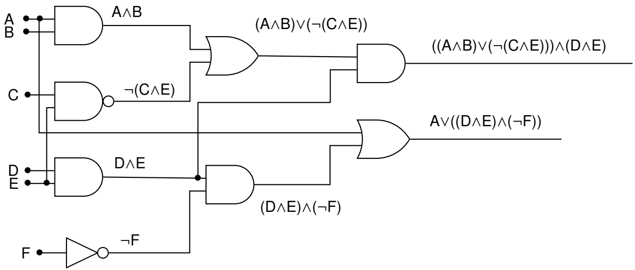
Example

Let S be a set, then $\mathcal{P}(S)$ is the powerset of S (the set of all subsets of S). $\mathcal{P}(S)$ has the structure of a partial order. The order relation is $\subseteq$. That is, there is a morphism from one subset of S to another if the first one is a subset of the second. The set $\mathcal{P}(S)$ is a partial order (and a Boolean algebra, which we will meet in Section **??***) and hence a category.*

We have seen that every monoid, every set, and every partial order forms a category. What about a graph? A graph does not necessarily have the structure of a category. A graph might not have identity morphisms and a graph by itself does not have a composition operation. So while the category of all graphs and graph homomorphisms forms the category $\mathbb{Graph}$, each individual graph need not form a category.
Moving on to some logic examples:

Example

Examples from Computers



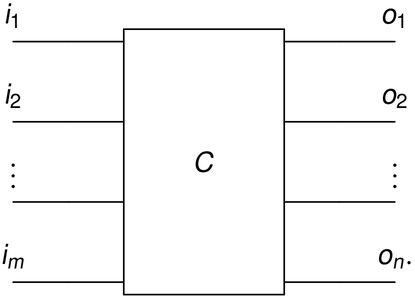
A logical circuit with 6 inputs and 2 outputs.

Examples from Computers

Let us just remember some simple logical gates:
These are the AND, OR, NOT, NAND and NOR gates. The AND, OR, XOR, NAND gates are all elements of $Hom(2, 1)$ and the NOT gate is in $Hom(1, 1)$

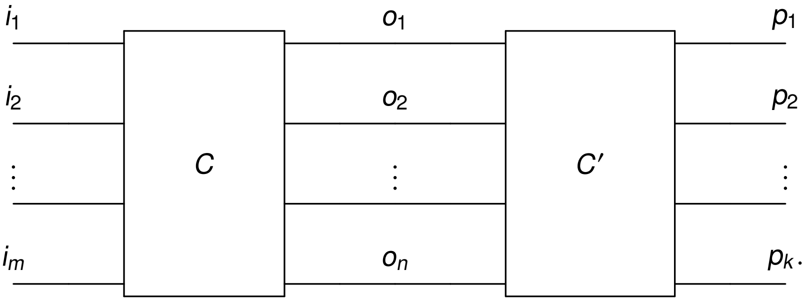
Examples from Computers

Let us draw a circuit with m input wires and n output wires as



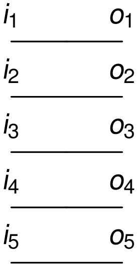
Examples from Computers

The composition of two circuits is as follows. Given two circuits, C with m input wires and n output wires, and C' with n input wires and k output wires, we can compose them to form $C' \circ C$ as



which has m input wires and k output wires.
The composition is obviously associative.

For each n, the identity circuit is simply n straight wires that do nothing. For example Id_5 looks like this



Examples from Logic

Let us close our first list of examples with the notion of proof.

Example

For this example we will have to fix some type of logical system (we are being intentionally vague). For each such logical system there will be a category $\mathbb{Proof}$. The objects of the category will be formal and exact logical statements, just like in $\mathbb{Prop}$. A morphism from A to B will be a formal and exact proof that assumes A is true and concludes that B is true. If there are formal proofs $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ then by concatenating the two proofs there is a formal proof $g \circ f\colon A \longrightarrow C$. The proof with just the statement A goes from A to A which corresponds to the identity on A. Notice that this category has the same objects as $\mathbb{Prop}$ but in contrast to $\mathbb{Prop}$ where there is at most one morphism between any two objects, here in $\mathbb{Proof}$, there might be more than one proof between logical statements and hence more than one morphism between two objects.

Examples from Logic

Now that we finished our first batch of examples, we need to reflect on some general ideas about the size of categories. Set theorists make a distinction between collections called “classes” and collections called “sets”. A class is a very “large” collection of entities while a set is a somewhat “small” collection of entities that are part of a class. Mathematicians usually restrict themselves to work with sets rather than classes. One talks about sets with algebraic structure or sets with topological structure. One does not hear about classes with such structures. This text will not worry much about these issues. However, definitions are in order.

Definition

A category is called **small** *if the objects and morphisms of the category are sets and not classes. If either the objects or the morphisms are a class, the category is called* **large***. A category is called* **locally small** *if each of the Hom sets are sets and not classes.*

Let us discuss sizes of some categories we have seen. Categories that are small are partial orders, preorders, and groups or monoids as single-object categories. Most of the rest of the categories that we dealt with are locally small but not small. For example, the category of sets is locally small but not small because the objects, which is the collection of all sets, is a class and not a set. Notice that even the subcollection of one-element sets is a class and not a set. The categories $\mathbb{Graph}$ and $\mathbb{Group}$ each have a proper class of objects but their Hom sets are proper sets. These categories are locally small. For the same reason, $\mathbb{Top}$ and $\mathbb{Manif}$ are locally small.

Examples from Logic

Before closing this section, it is worth noting that there is another definition of a category which is equivalent to Definition **??**. Equivalent means that these two definitions describe the same structures. The main difference is that while the previous definition uses a collection of objects and a collection of morphisms, this definition only uses a collection of morphisms (i.e., it is a "single-sorted theory"). Objects are not mentioned in this definition but they are essentially associated with special types of morphisms called identity morphisms. The domain and codomain of every morphism are such identity morphisms. In other words, we can view a function f as

$$\begin{tikzcd} dom(f) \arrow[loop right] \arrow[rrr, "f"] & & & cod(f) \arrow[loop left] \end{tikzcd}$$

We did not start with this definition as it is slightly less concrete and might discourage the novice. We mention it now because this definition is actually simpler (single-sorted rather than two-sorted) and it will be useful when talking about higher-dimensional categories (in particular see Definition **??** in Section **??** .) The

Another Definition of a Category

Definition

A **category** $\mathbb{A}$ *is a collection of morphisms* $Mor(\mathbb{A})$ *with the following structure:*

- *Every morphism has a domain morphism: there is a function* $dom\colon Mor(\mathbb{A}) \longrightarrow Mor(\mathbb{A})$.
- *Every morphism has a codomain morphism: there is a function* $cod\colon Mor(\mathbb{A}) \longrightarrow Mor(\mathbb{A})$.
- *There is a composition operation* $\circ$*: if* f *and* g *satisfy* $cod_{\mathbb{A}}(f) = dom_{\mathbb{A}}(g)$*, then there is an associated morphism* $g \circ f$.

Another Definition of a Category

Definition (Continued.)

These operations must satisfy the following axioms:

- *The domain of a composite is the domain of the first:* $dom(g \circ f) = dom(f)$.
- *The codomain of a composite is the codomain of the second:* $cod(g \circ f) = cod(g)$.
- *The composition is associative:* $h \circ (g \circ f) = (h \circ g) \circ f$.
- *The domain morphism acts like an identity:* $f \circ dom(f) = f$.
- *The codomain morphism acts like an identity:* $cod(f) \circ f = f$.
- *The domain morphisms come and go from themselves:* $dom(dom(f)) = dom(f) = cod(dom(f))$.
- *The codomain morphisms come and go from themselves:* $dom(cod(f)) = cod(f) = cod(cod(f))$.

Foreshadowing

- Chapter 2: Categories
  - Section 2.2: Basic Properties
    - Commutative Diagrams
    - Types of Morphisms
    - Types of Objects
    - Uniqueness of Objects and Morphisms

Definition

A **diagram** *is a part of a category that has objects and morphisms between those objects. We say that a diagram* **commutes** *or is a* **commutative diagram** *if any two paths from the same starting object to the same finishing object actually describe the same morphism in the category.*

Basic Properties of Categories

Three examples of commutative diagrams

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \\ c \arrow[r, "i"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "h"] & c \end{tikzcd}$$

(i) (ii) (iii)

- (i) is a commutative triangle and it says that $g \circ f = h$.
- (ii) is commutative square and it says that $g \circ f = i \circ h$.
- (iii) means that $h \circ f = h \circ g$.

Basic Properties of Categories

As another example, we can write the requirement that morphism composition has to be associative and that composition with the identity morphisms does not change the morphism as the following two commutative diagrams:

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] \arrow[rrr, bend left=60, "h \circ (g \circ f)"] \arrow[rrr, bend right=60, "(h \circ g) \circ f"'] & b \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & c \arrow[r, "h"] & d
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "id_a"] \arrow[rr, bend left, "f"] & a \arrow[r, "f"] \arrow[rr, bend right, "f"'] & b \arrow[r, "id_b"] & b
\end{tikzcd}
$$

Special Morphisms of Categories

Within a category, there are special types of morphisms.

Definition

A morphism $f: a \longrightarrow b$ is a **monomorphism** *or* **monic** *if for all objects c and for all $g: c \longrightarrow a$ and $h: c \longrightarrow a$ with the relationship*

$$\begin{tikzcd} c \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

the following rule is satisfied:

$$\text{If } f \circ g = f \circ h, \text{ then } g = h.$$

Another way to say that f is monic is to say that f is **left cancelable**. *That is, if f is on the left of two sides of an equation, we can cancel it.*

Example

In $\mathbb{Set}$, a map $f\colon S \longrightarrow T$ is monic if and only if it is an injection (one-to-one). Assume f is monic. Consider the one-object set $\{*\}$. Let $g\colon \{*\} \longrightarrow S$ and $h\colon \{*\} \longrightarrow S$ each pick out an element of S. The statement that f is monic amounts to

$$\text{If } f(g(*)) = f(h(*)), \text{ then } g(*) = h(*).$$

Thinking of $g(*)$ and $h(*)$ as elements in S, this is exactly the requirement that f is an injection. To go the other way, remember we saw that f is injective if and only if there is a f' such that $f' \circ f = id_a$. So if $f \circ g = f \circ h$, then we can compose both sides with f' to get $f' \circ f \circ g = f' \circ f \circ h$ and hence $g = h$.

Exercise

Show that the composition of two monics is monic.

Special Morphisms of Categories

Definition

A morphism $f\colon a \longrightarrow b$ is an **epimorphism** *or* **epic** *if for all objects c and for all $g\colon b \longrightarrow c$ and $h\colon b \longrightarrow c$ with the relationship*

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & c \end{tikzcd}$$

then the following rule is satisfied:

$$\textit{If } g \circ f = h \circ f, \textit{ then } g = h.$$

Another way to say that f is epic is to say that f is **right cancelable**.

Exercise

Show that the composition of two epics is epic.

Special Morphisms of Categories

Definition

A morphism $f\colon a \longrightarrow b$ is an **isomorphism** *if there exists a morphism $g\colon b \longrightarrow a$ such that $g \circ f = id_a$ and $f \circ g = id_b$. The morphisms f and g are called* **inverses** *of each other. In such a situation we say that a and b are* **isomorphic** *and we write this as $a \cong b$.*

Special Morphisms of Categories

Exercise

Show that if a morphism has an inverse it is unique.

Exercise

Show that any identity morphism is a isomorphism.

Exercise

Show that the composition of two isomorphisms is an isomorphism.

Exercise

Show that in any category the relation of being isomorphic is an equivalence relation.

Notice that although the usual definitions of injective, surjective, and isomorphism are given in terms of elements, the definitions of epic, monic, and isomorphism are given here in terms of morphisms in a category. This is of fundamental importance in category theory.

There are two types of categories with only isomorphisms.

Definition

A **groupoid** *is a category where all the morphisms are isomorphisms.*
A **group** *is a type of groupoid which has only one object.*

Special Morphisms of Categories

Example

- *Not only does the entire collection of groups form a category, but each individual group can be seen as a special type of category.*
- *If G is a group then $C(G)$ is a category with one object and the morphisms from the single object to itself are the elements of G.*
- *The identity in the group becomes the identity of the category and composition of the morphisms is the group multiplication.*
- *Associativity of the group multiplication becomes associativity of morphism composition. The fact that every element of the group has an inverse is another way of saying that every morphism in the category is an isomorphism.*
- *Yet another way to describe a group is to say a group is a one-object category where every morphism is invertible.*

Special Objects of Categories

Let us move on and talk about special types of objects in a category. There are certain objects that can be described by the way they relate with the morphisms in the category.

Definition

- *An object i in a category $\mathbb{A}$ is called an* **initial object** *if there is a unique morphism from i to every object in $\mathbb{A}$ (including i).*
- *An object t is called a* **terminal object** *if there is a unique morphism from every object (including t) to t.*
- *An object z is called a* **zero object** *if it is both an initial object and a terminal object.*

When we want to stress that a map $f\colon a \longrightarrow b$ uniquely satisfies a property, we will write it as $a \xrightarrow[\exists!]{f} b$ or $f\colon a \xrightarrow{!} b$.

Example

- *The empty set $\emptyset$ is the initial object in the category of $\mathbb{Set}$ because for any set S there is a unique function $f\colon \emptyset \longrightarrow S$ or* $\emptyset \xrightarrow[\exists!]{f} S$.
- *Any single element set $\{*\}$, $\{x\}$, or $\{Wanda\}$ is a terminal object in the category $\mathbb{Set}$ because for any set S there is a unique function $f\colon\ S \longrightarrow \{*\}$.*
- *The category $\mathbb{Set}$ does not have a zero object.*
- *Notice that there is only one initial object in $\mathbb{Set}$, namely the empty set. In contrast, there is a whole class of terminal objects in $\mathbb{Set}$. While they are all isomorphic, they are not equal to each other.*

Special Objects of Categories

Example

- *Let us consider the algebraic categories.*
- *The empty set and the trivial operations on the empty set satisfy all the requirements of being a magma and a semi-group.*
- *The empty set is the initial object in the categories* $\mathbb{M}\mathrm{agma}$ *and* $\mathbb{S}\mathrm{emi}\mathbb{G}\mathrm{p}$.
- *In contrast, all the other algebraic categories that we looked at have at least one constant, and hence, the empty set does not satisfy the requirements.*

Special Objects of Categories

Example

- *In $\mathbb{Prop}$ the initial object is the proposition that is always false which is denoted $\perp$.*
- *It is the initial object because if you start off with a falsehood, any proposition is a consequence.*
- *The terminal object is the proposition that is always true, denoted $\top$ because it is a consequence of any proposition.*
- *(There is usually no zero object unless the logical system is inconsistant and $\top = \perp$. In that case, the logical system is totally worthless.)*
- *The category $\mathbb{Proof}$ does not have an initial or terminal object because, in general, there is more than one way to prove an implication.*

Special Objects of Categories

Example

- *Let* $(P, \leq)$ *be a partial order and* $A(P, \leq)$ *be its associated category.*
- *The initial object of that category is the bottom element which is below every other element.*
- *(This is not to be confused with an "atom" which is an element that only has the bottom element below it. Think of an atom as the smallest element —above nothing— and other elements are made of atoms. There might be many atoms but there is at most one bottom.)*
- *The terminal object is the top element.*
- *If* P *is non-trivial then there is no zero-object.*

Notice we say “an” initial object and not “the” initial object. The reason for this is that there might be more than one initial object. However if there is more than one, then they are related in an interesting way.

Theorem

There is a unique isomorphism between any two initial objects.

Proof.

- Let i and i' be objects in $\mathbb{A}$ and assume they are each an initial object.
- This argument is summarized with the following commutative diagram:

$$\begin{tikzcd}
i \arrow[r, "f"] \arrow[rr, bend right, "id_i" near end, "\exists!"'] & i' \arrow[r, "g"] \arrow[rr, bend left, "id_{i'}", "\exists!"'] & i \arrow[r, "f"] & i' .
\end{tikzcd}$$

- From the fact that i is initial and i' is an object in $\mathbb{A}$, there is a unique morphism $f\colon i \longrightarrow i'$.
- From the fact that i' is initial and i is an object in $\mathbb{A}$, there is a unique morphism $g\colon i' \longrightarrow i$.

Continued.

- From the fact that i is an initial object and i is also an object, there is a unique morphism from i to i which we know is the $id_i\colon i \longrightarrow i$.
- Since there is no other morphism from i to i, it must be that $g \circ f = id_i$.
- From the fact that i' is an initial object, $id_{i'}\colon i' \longrightarrow i'$ is the unique morphism from i' to i' which means $f \circ g = id_{i'}$.
- This shows that $g = f^{-1}$ and $f = g^{-1}$.
- This isomorphism $i \longrightarrow i'$ is unique because another isomorphism would entail a violation of i being initial.

$\square$

This type of theorem — which shows how unique an object or a morphism is — arises frequently in category theory. We say that the initial object is "unique up to a unique isomorphism."

Central Idea

Important Categorical Idea

The Uniqueness of Morphisms.

- *We will be very concerned with how unique an entity is.*
- *Sometimes we will describe an entity by giving its requirements and there will only be one entity that satisfies the requirements. (For example, in the category of sets there is only one empty set.)*
- *Sometimes many entities will satisfy the requirements and all those entities are isomorphic to each other. (For example, in the category of sets, there are many sets with three objects, e.g., $\{a, b, c\}$, $\{x, y, z\}$ etc. All these sets are isomorphic to each other, however, there are six possible isomorphisms between any two such sets.)*

Important Categorical Idea

The Uniqueness of Morphisms.

- *There is an intermediate level: there are some requirements that are satisfied by many different entities, and all these different entities are isomorphic to each other, but there is a unique isomorphism between them. (For example, in the category of sets, there are many one-element sets, and they are all isomorphic to each other with a unique isomorphism between them.)*

We can summarize this hierarchy as follows:

- *unique.*
- *unique up to a unique isomorphism.*
- *unique up to an isomorphism.*

Exercise

Prove that there is a unique isomorphism between any two terminal objects.

Let us look at some more examples of initial objects, terminal objects, and zero objects in a category.

Example

- $\mathbb{Top}, \mathbb{Manif}$: *The empty set is the initial object and the single point is a terminal object. There are no zero objects.*
- $\mathbb{Magma}, \mathbb{SemiGp}$: *The empty set is the initial object. The single-element structure is the terminal object. There are no zero objects.*
- $\mathbb{Monoid}, \mathbb{Group}, \mathbb{AbGp}, \mathbf{K}\mathbb{Vect}$: *The single-element structure is the zero object.*
- $\mathbb{Ring}$: *There is a one-object ring where* $0 = 1$ *and this ring is terminal in the category. The ring of* $\mathbf{Z}$ *is initial in the category.*

Special Objects of Categories

Example (Continued.)

- $\mathbb{Field}$: *The single element set is not an object in this category because we need two different constants (0 and 1). However, this two-element field is neither initial nor terminal.*
- $\mathbf{K}\mathbb{Mat}$: 0 *is the zero object.*
- $\mathbb{PO}, \mathbb{PreO}$: *The empty set is the initial object. A one-element set is a terminal object.*

Foreshadowing

- Chapter 1: Categories
  - Section 2.3: Related Categories
    - Subcategories
    - Quotient Categories
    - Skeletal Categories
    - Opposite Categories
    - Cartesian Product of Categories.

Subcategories

Just as there is an obvious notion of a subset of a set, so too there is an obvious notion of a subcategory.

Definition

Category $\mathbb{A}$ is a **subcategory** *of category $\mathbb{B}$ (or, another way to say this is that category $\mathbb{B}$ is a* **supercategory** *of $\mathbb{A}$) if*

- *The objects of $\mathbb{A}$ are part of the objects of $\mathbb{B}$: $Ob(\mathbb{A}) \subseteq Ob(\mathbb{B})$.*
- *The morphisms of $\mathbb{A}$ are a part of the morphisms of $\mathbb{B}$: $Mor(\mathbb{A}) \subseteq Mor(\mathbb{B})$.*
- *The composition operation for $\mathbb{A}$ is the same as the composition operation in $\mathbb{B}$ but restricted to the elements of $\mathbb{A}$.*
- *The identity morphisms of $\mathbb{A}$ are the same as the identity morphisms of $\mathbb{B}$.*

Subcategories

Example

The category $\mathbb{F}\mathrm{in}\mathbb{S}\mathrm{et}$ of all finite sets and set functions between them is a subcategory of $\mathbb{S}\mathrm{et}$.

Example

The category $\mathbf{N}\mathbb{M}\mathrm{at}$ of matrices with natural number entries is a subcategory of $\mathbf{Z}\mathbb{M}\mathrm{at}$ where the entries are integers. Notice that the objects of both of these two categories are the set of natural numbers. Since

$$\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$$

we have that $\mathbf{Z}\mathbb{M}\mathrm{at}$ is a subcategory of $\mathbf{Q}\mathbb{M}\mathrm{at}$ which in turn is a subcategory of $\mathbf{R}\mathbb{M}\mathrm{at}$ which in turn is a subcategory of $\mathbf{C}\mathbb{M}\mathrm{at}$.

Subcategories

Example

- *Consider the category* $\mathbb{NANDCircuit}$.
- *The objects are the natural numbers, and the set of morphisms from m to n is the set of all logical circuits made of NAND gates that have m input wires and n output wires.*
- *This category is a subcategory of* $\mathbb{Circuit}$.
- *Notice that* $\mathbb{NANDCircuit}$ *and* $\mathbb{Circuit}$ *have the same objects.*
- *It is also well known that every logical circuit with different types of gates can mimicked by a logical circuit with only NAND gates. This means that every morphism in* $\mathbb{Circuit}$ *has at least one corresponding morphism in* $\mathbb{NANDCircuit}$.

Subcategories

Example

The category $\mathbb{AbGp}$ *is a subcategory of* $\mathbb{Group}$. *The category of* $\mathbb{SemiGp}$ *is a subcategory of* $\mathbb{Magma}$. *There are many other such subcategories when dealing with algebraic structures.*

Example

Since every partial order (reflexive, transitive, and anti-symmetric) is a preorder (reflexive and symmetric), and the morphisms between partial orders and preorders are order preserving maps, the category of partial orders $\mathbb{PO}$ *is a subcategory of* $\mathbb{PreO}$.

Example

Every category has a subcategory that is a groupoid. Just take all the morphisms that are isomorphisms. Since every identity morphism is an isomorphism, the groupoid subcategory has the same objects as the category.

Quotient Categories

Given a set and an equivalence relation on the set, one can construct a quotient set. Similarly, given a category and a souped-up equivalence relation, there is a notion of a quotient category. The souped-up equivalence relation is a relation that respects the composition operation in the category.

Definition

Let $\mathbb{A}$ be a category. A **congruence relation** *or* **congruence** *on $\mathbb{A}$ is an equivalence relation on the collection of morphisms of $\mathbb{A}$ that respects the composition. In detail, a congruence $\sim$ is an equivalence relation on each of the Hom sets of the category which satisfy the following requirement: let $f, f' : a \longrightarrow b$ and $g, g' : b \longrightarrow c$ then the rule*

$$\text{if } f \sim f' \text{ and } g \sim g', \text{ then } (g \circ f) \sim (g' \circ f')$$

is satisfied.

Quotient Categories

With the notion of a congruence, we go on to define a quotient category.

Definition

Let $\mathbb{A}$ be a category and $\sim$ be a congruence relation on $\mathbb{A}$, then a **quotient category** $\mathbb{A}/_\sim$ *is a category constructed as follows: The objects of $\mathbb{A}/_\sim$ are the same as $\mathbb{A}$ and the morphisms of $\mathbb{A}/_\sim$ are the quotient collection of morphisms of $\mathbb{A}$ under the congruence relation $\sim$. This means that for any objects a and b, we have*

$$Hom_{\mathbb{A}/_\sim}(a,b) = Hom_{\mathbb{A}}(a,b)/_\sim.$$

The composition operation in $\mathbb{A}/_\sim$ follows from the composition operation of $\mathbb{A}$. In detail, $[g] \circ [f] = [g \circ f]$.

Other Related Categories

Usually in a category, there are a lot of objects that are isomorphic but not equal to each other. There are special categories where there are no isomorphisms between different objects.

Definition

A category is **skeletal** *if any two isomorphic objects are equal.*

Other Related Categories

Every category $\mathbb{A}$ has an associated skeletal category $sk(\mathbb{A})$. The objects of $sk(\mathbb{A})$ are a collection of objects of $\mathbb{A}$ where each object of $\mathbb{A}$ is isomorphic to some object in $sk(\mathbb{A})$. In other words, every object in the original category has an isomorphic representation in the skeletal category. The skeletal category is a subcategory of the original category. (This construction demands the axiom of choice.)

Other Related Categories

The following example goes back to the spirit of the philosopher Gotlieb Frege. He defined a natural number as the equivalence class of all finite sets with that number of elements.

Example

Consider the category of $\mathbb{FinSet}$. A skeletal category for $\mathbb{FinSet}$ is Nat$\mathbb{Set}$. The objects are the empty set and the sets $\{1\}, \{1,2\}, \{1,2,3\}, \ldots$. Notice that none of these objects are isomorphic to any other. Every finite set is isomorphic to one of these sets. The morphisms in Nat$\mathbb{Set}$ are all functions between these sets.

Other Related Categories

Example

Let $(P, \leq)$ be a preorder category. A skeletal category of $(P, \leq)$ is a partial order category. In detail, we form a partial order category $(P_0, \leq)$ where the objects of P_0 are representatives of isomorphism classes of P. Two representatives p and p' have the relation $p \leq_0 p'$ if $p \leq p'$.

Other Related Categories

Definition

For a category $\mathbb{A}$, the **opposite category** $\mathbb{A}^{op}$ *is a category with the same objects as $\mathbb{A}$ but with the domain and codomain of each arrow of $\mathbb{A}$ reversed. Every morphism $a \longrightarrow b$ in $\mathbb{A}^{op}$ corresponds to a morphism $b \longrightarrow a$ in $\mathbb{A}$ or $Hom_{\mathbb{A}^{op}}(a, b) = Hom_{\mathbb{A}}(b, a)$. The composition in $\mathbb{A}^{op}$ is given by the composition in $\mathbb{A}$. In detail, if*

$$a \xrightarrow{\quad f \quad} b \xrightarrow{\quad g \quad} c$$

are two composible morphisms in $\mathbb{A}^{op}$, then there are two composible morphisms in $\mathbb{A}$

$$a \xleftarrow{\quad f' \quad} b \xleftarrow{\quad g' \quad} c.$$

The composition $g \circ f$ in $\mathbb{A}^{op}$ will correspond to the composition $f' \circ g'$ in $\mathbb{A}$.

Example

$\mathbb{Rel}^{op}$ is almost the same as $\mathbb{Rel}$. (We will formulate what we mean by "almost the same" later.) The objects are all sets. A relation $R \subseteq S \times T$ is associated to the inverse relation $R^{-1} \subseteq T \times S$. Notice that this statement is not true for $\mathbb{Set}$ and $\mathbb{Par}$ because not every function or partial function has an inverse.

Example

If $(P, \leq)$ is a preorder category then the opposite category has the relation $\leq^{op}$ which is defined as $p \leq^{op} p'$ if and only if $p' \leq p$. This essentially turns the arrows around.

Other Related Categories

Example

The category $\mathbf{K}\mathbb{M}\mathbb{a}\mathbb{t}$ *has natural numbers as objects and a map* $A\colon m \longrightarrow n$ *is a* n *by* m *matrix with entries in* $\mathbf{K}$*. The category* $\mathbf{K}\mathbb{M}\mathbb{a}\mathbb{t}^{op}$ *has natural numbers as objects and a map* $A\colon m \longrightarrow n$ *is a* m *by* n *matrix with entries in* $\mathbf{K}$*. Every matrix* $A\colon m \longrightarrow n$ *in* $\mathbf{K}\mathbb{M}\mathbb{a}\mathbb{t}$ *will correspond to the transpose matrix* $A^T\colon n \longrightarrow m$ *in* $\mathbf{K}\mathbb{M}\mathbb{a}\mathbb{t}^{op}$*.*

Exercise

Show that $f\colon a \longrightarrow b$ *in* $\mathbb{A}$ *is monic if and only if the corresponding* $g\colon b \longrightarrow a$ *in* $\mathbb{A}^{op}$ *is epic.*

Exercise

Show that $\mathbb{A}$ *has an initial object if and only if* $\mathbb{A}^{op}$ *has a terminal object.*

Other Related Categories

There is an important operation on categories. Just as we can take two sets and form their Cartesian product, so we can take two categories and form their Cartesian product.

Definition

Given categories $\mathbb{A}$ and $\mathbb{B}$, there is a category which is called the **Cartesian product** *of $\mathbb{A}$ and $\mathbb{B}$, written as $\mathbb{A} \times \mathbb{B}$. The objects are pairs of objects (a, b) where a is an object of $\mathbb{A}$ and b is an object of $\mathbb{B}$. Morphisms in the category are pairs of morphisms $(f, g) \colon (a, b) \longrightarrow (a', b')$ where $f \colon a \longrightarrow a'$ is in $\mathbb{A}$ and $g \colon b \longrightarrow b'$ is in $\mathbb{B}$. Composition is given "component-wise", i.e., the composition of $(f, g) \colon (a, b) \longrightarrow (a', b')$ and $(f', g') \colon (a', b') \longrightarrow (a'', b'')$ is*

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g) \colon (a, b) \longrightarrow (a'', b'').$$

(This identity is an instance of the "interchange law" which will be discussed in Important Categorical Idea **??***.) We leave it to the readers to prove associativity of composition. The unit for (a, b) is*

Foreshadowing

- Chapter 2: Categories
  - Section 2:4: Mini-course: Basic Linear Algebra
    - The Objects: Vector Spaces
    - The Morphisms: Linear Transformations
    - Bases and Dimensions
    - Operations on Vector Spaces
- We define vector spaces
- We define linear transformations between vector spaces
- Lots of examples and properties
- How to make new vector spaces from old
- Our focus is — as always — the morphisms, linear transformations.

Foreshadowing

Linear algebra is the study of directions, straight lines, and flat spaces. The mathematical structures that describe such notions are vector spaces. Functions that go from one vector space to another are called linear transformations. The subject of linear algebra are vector spaces and linear transformations. The collection of vector spaces and linear transformations forms the category $\mathbb{Vect}$. This mini-course will explore the category of $\mathbb{Vect}$ with special emphasis on the morphisms.

We are going to focus on complex vector spaces, $\mathbf{C}\mathbb{Vect}$, which are vector spaces associated to complex numbers. We require such vector spaces later when we talk about quantum mechanics and many of the mini-courses.

A small disclaimer. Linear algebra is an important field that is associated with beautiful geometric ideas. There is no way we can show you anything more than a birds-eye-view of the field. We hope that this mini-course will whet your appetite to learn more linear algebra.

The Objects: Vector Spaces

The objects of $\mathbf{C}\mathbb{Vect}$ are complex vector spaces.

Definition

Consider **C**, *the field of complex numbers. A* **complex vector space** *is an abelian group* $(V, +, 0, -)$ *(reminder: V is a set, $+$ is an associative, commutative binary operation with identity 0 and inverse operation $-$) with a function* $\cdot : \mathbf{C} \times V \longrightarrow V$ *which is called* **scalar multiplication** *(or a* **C-action***) on V. (A scalar is an element of a field or simply a number.) The elements of V will be called "vectors" and will correspond to the directions. The $+$ and $-$ operations allow us to add and subtract directions. The scalar multiplication operation permit us to lengthen or shorten the directions. The $\cdot$ function must satisfy the following axioms.*

- *The scalar multiplication respects the addition in the abelian group: for all c in* **C** *and for all v, v' in V,*

$$c \cdot (v + v') = (c \cdot v) + (c \cdot v').$$

The Objects: Vector Spaces

Example

There are many examples of complex vector spaces.

- *For any positive integers m and n, the set of m by n matrices with elements in the field* $\mathbf{C}$ *has the structure of a vector space. We denote this complex vector space as* $\mathbf{C}^{m\times n}$. *The addition in this vector space is simply addition of matrices. The negation operation multiplies every entry by* -1. *The zero vector is the m by n matrix whose entries are all 0. Given a matrix M and a scalar* $c \in \mathbf{C}$, *the scalar multiplication* $c \cdot M$ *is the matrix M with all its entries multiplied by c. (Another way of saying this is that for every m and n, the Hom set,* $Hom_{\mathbf{C}\mathbb{M}\mathrm{at}}(m,n)$, *is a complex vector space.)*
- *In particular, if* $n = 1$, *then* $\mathbf{C}^{m\times 1} = \mathbf{C}^m$ *is the set of all m-element column vectors. This vector space shares the same operations as* $\mathbf{C}^{m\times n}$. *Similarly, if* $m = 1$ *then* $\mathbf{C}^{1\times n}$ *is the set of n-element row vectors. These examples are the core of elementary linear algebra.*

Exercise

Let S be any set. Show that the set of all functions from S to $\mathbf{C}$ forms a complex vector space.

Central Idea

Important Categorical Idea

Hom Sets Inheriting Properties.

- *In a category $\mathbb{A}$, if a and b are objects in $\mathbb{A}$, then $Hom_{\mathbb{A}}(a, b)$ will inherit many of the properties of b.*
- *This simple fact arises over and over again.*

The Objects: Vector Spaces

Just as we can look at subsets of a set, we can look at subvector spaces of a vector space.

Definition

A vector space W is a **linear subspace** *or* **subvector space** *of vector space V if W is a subset of V and*

- *W is closed under addition: for all w, w' in W, $w + w'$ is in W.*
- *W is closed under scalar multiplication: for all c in $\mathbf{C}$ and w in W, $c \cdot w$ is in W.*
- *W has the zero vector: 0 is in W.*

The Objects: Vector Spaces

Example

Here are some examples of linear subspaces.

- *The trivial vector space,* 0*, is a linear subspace of every vector space.*
- *It is not hard to see that* $\mathbf{C}$ *is a linear subspace of* $\mathbf{C}^m$ *for any* $m > 1$ *(identify* $c \in \mathbf{C}$ *as* $\begin{bmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$*.) Similarly,* $\mathbf{C}^m$ *is a linear*

$$\begin{bmatrix} c_1 \\ c_2 \\ \end{bmatrix} \quad \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 \\ c_2 & 0 & 0 & \cdots & 0 \\ \end{bmatrix}$$

As we have seen (Important Categorical Idea), in category theory the morphisms are as important or even more important than the objects.

Definition

Let V and W be vector spaces. A **linear map** *or* **linear transformation** *T from V to W, written $T\colon V \longrightarrow W$ is a set function $T\colon V \longrightarrow W$ such that*

- *T respects the addition operation: for all $v, v' \in V$,*

  $$T(v + v') = T(v) + T(v').$$

  (Notice that the two $+$'s are operations in two different vector spaces. The one on the left is an operation in V and the one on the right is an operation in W.)

- *T respects the scalar multiplication: for all $c \in \mathbf{C}$ and $v \in V$,*

  $$T(c \cdot v) = c \cdot T(v).$$

Example

A few examples of linear maps.

- *If W is a linear subspace of V, then the inclusion map is a linear transformation.*
- *For vector spaces V and W, there is a simple linear transformation, called the* **null map** *or* **zero map**, $N\colon V \longrightarrow W$ *that takes every element of V to the 0 in W. That is, $N(v) = 0$.*
- *If A is an $m \times n$ matrix with entries in $\mathbf{C}$, then there is a linear map $T_A : \mathbf{C}^n \longrightarrow \mathbf{C}^m$ defined for $V \in \mathbf{C}^n$ as $T_A(V) = AV$. This is linear because*
  - *T_A respects the matrix addition:*

    $$T_A(V + V') = A(V + V') = AV + AV' = T_A(V) + T_A(V'),$$

    and
  - *T_A respects the scalar multiplication:*

    $$T_A(c \cdot V) = A(c \cdot V) = c \cdot (AV) = c \cdot T_A(V).$$

- *There is a linear transformation from $\mathbf{C}^{m\times n}$ to $\mathbf{C}^{n\times m}$ that is*

Theorem

Let $T\colon V \longrightarrow W$ be a linear map. Then T respects 0, i.e., $T(0) = 0$. Also T respects negation, i.e., $T(-v) = -T(v)$.

Proof.

To show that T respects 0, notice that

$$T(0) = T(0+0) = T(0) + T(0).$$

Now subtract $T(0)$ from both ends. This leaves $0 = T(0)$.
To show that T respects negation, notice that

$$0 = T(0) = T(v + (-v)) = T(v) + T(-v).$$

Now subtract $T(v)$ from both ends of the equation. This leaves $-T(v) = T(-v)$. $\square$

Exercise

Show that the composition of two linear maps is a linear map. Moreover, show that this composition is associative.

Exercise

Show that for every vector space V, the identity function $I_V \colon V \longrightarrow V$, which is defined for $v \in V$ as $I_V(v) = v$, is a linear map. Show also that for $T \colon V \longrightarrow V'$, we have that $T \circ I_V = T = I_{V'} \circ T$.

Complex Vector Spaces

The previous two exercises bring us to the main definition of this mini-course:

Definition

The collection of complex vector spaces and linear maps between them forms a category $\mathbf{C}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}$.

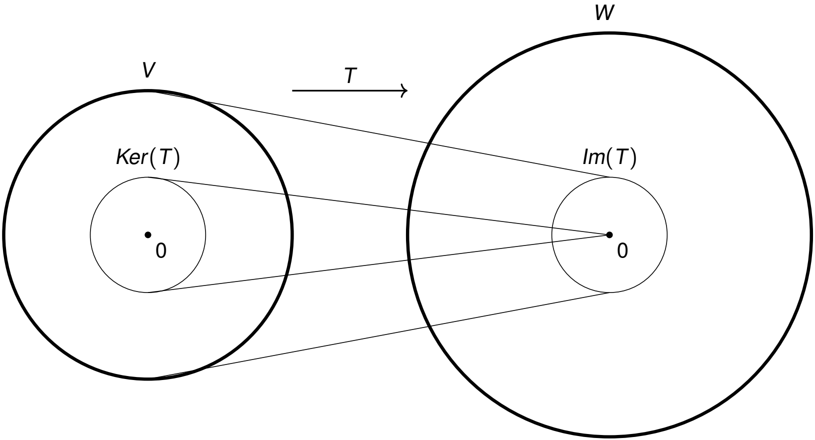
The Objects: Vector Spaces

Remark

In the category $\mathbb{Set}$, an element of a set S is determined by a map from a terminal object to S, i.e., a map $\{*\} \longrightarrow S$. In the category $\mathbf{C}\mathbb{Vect}$, a vector of a vector space V is determined by a linear map from $\mathbf{C}$ to V. (A map from the terminal object 0 in $\mathbf{C}\mathbb{Vect}$ to V, i.e., $0 \longrightarrow V$ just picks out the 0 vector in V, which is not very helpful.) Any map $f\colon \mathbf{C} \longrightarrow V$ is totally determined by where it takes $1 \in \mathbf{C}$. In other words, f is defined by $f(1) \in \mathbf{C}$. For any $c \in \mathbf{C}$, the value $f(c) = f(c \cdot 1) = c \cdot f(1)$. In other words, $Hom_{\mathbf{C}\mathbb{Vect}}(\mathbf{C}, V)$ is isomorphic to V. However, the output of f is not only $f(1)$. Rather, the output is all the scalar multiples of $f(1)$. In particular, thinking of $\mathbf{C}$ as a complex vector space, we have that all linear maps $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ is determined by the value $f(1)$. In other words $Hom_{\mathbf{C}\mathbb{Vect}}(\mathbf{C}, \mathbf{C})$ is isomorphic to $\mathbf{C}$.

The kernel and image of a linear transformation

For any linear map $T\colon V \longrightarrow W$, there is a linear subspace of V, the kernel of T, and a linear subspace of W, the image of T



Kernels and Images

Definition

For linear map $T \colon V \longrightarrow W$, the **kernel** *of T, written $Ker(T)$, is the linear subspace of V consisting of those vectors that go to the zero vector of W, i.e.,*

$$Ker(T) = \{v \in V : T(v) = 0\}.$$

To see that the $Ker(T)$ is a linear subspace of V, notice that

- *$Ker(T)$ is closed under addition: if $T(v) = 0$ and $T(v') = 0$, then $T(v + v') = T(v) + T(v') = 0 + 0 = 0$,*
- *$Ker(T)$ is closed under scalar multiplication: if c is in* **C** *and $T(v) = 0$, then $T(c \cdot v) = c \cdot T(v) = c \cdot 0 = 0$, and*
- *$Ker(T)$ contains the zero vector: $T(0) = 0$.*

Kernels and Images

Definition

For linear map $T\colon V \longrightarrow W$, the **image** *of T, written $Im(T)$, is the linear subspace of W consisting of those vectors that are the output of T, i.e.,*

$$Im(T) = \{w \in W : \text{there exists a } v \in V \text{ with } T(v) = w\}.$$

To see that the $Im(T)$ is a linear subspace of W, notice that

- *$Im(T)$ is closed under addition: if $T(v) = w$ and $T(v') = w'$, then $w + w' = T(v) + T(v') = T(v + v')$,*
- *$Im(T)$ is closed under scalar multiplication: if c is in* **C** *and $T(v) = w$, then $c \cdot w = c \cdot T(v) = T(c \cdot v)$, and*
- *$Im(T)$ contains the zero vector: $T(0) = 0$.*

Kernels and Images

Theorem

For a linear map $T\colon V \longrightarrow W$, the following are equivalent: (i) T is monic, (ii) T is an injection, and (iii) $Ker(T) = \{0\}$.

Proof.

The fact that monic is equivalent to injective is basically the same proof as in $\mathbb{Set}$. Rather then use the one-object set $\{*\}$, use the one-dimensional vector space **C**. If $T\colon V \longrightarrow W$ is an injection, then since $T(0) = 0$, it is the only vector that goes to 0. Hence $Ker(T) = \{0\}$. For the other way, assume that $Ker(T) = \{0\}$ and that $T(x) = T(y)$. Then $0 = T(x) - T(y) = T(x - y)$. This means that $x - y$ is in the kernel of T. That means $x - y = 0$ and hence $x = y$. □

Linear Maps

Theorem

A linear map $T\colon V \longrightarrow W$ is epic if and only if T is a surjection.

Proof.

This is basically the same proof as in $\mathbb{Set}$. Rather then use the one-object set $\{*\}$, use the one-dimensional vector space **C**. $\square$

Isomorphism

Definition

A linear map is an **isomorphism** *if it is has an inverse. If there is an isomorphism between two vector spaces, then the vector spaces are called* **isomorphic***.*

Isomorphism

Theorem

A linear map is an **isomorphism** *if and only if it is monic and epic.*

Proof.

If a linear map is an isomorphism, then use the inverse to show that it is left and right cancelable. If it is monic then it is injective. If it is epic then it is surjective. So there is a set theoretic inverse to the map. It remains to show that inverse is linear. This follows from the fact that the map is linear and its inverse "undoes" what the map does. □

Linear Combination

Given a set of elements of V, we can use the addition and scalar multiplication operations to get other elements. (For convenience, we will drop the $\cdot$ as the scalar multiplication and write cx for $c \cdot x$.)

Definition

If $x_1, x_2, x_3, \ldots$ are vectors in a vector space V and $c_1, c_2, c_3, \ldots$ are elements of $\mathbf{C}$ then

$$y = c_1x_1 + c_2x_2 + c_3x_3 + \cdots .$$

is a vector of V and is called a **linear combination** *of $x_1, x_2, x_3, \ldots$.*

Basis

Definition

A **basis** $B = \{b_1, b_2, b_3, \ldots\}$ *for a vector space* V *is a set of vectors of* V*, such that every element of* V *can be written in exactly one way as a linear combination of elements of* B*. That is, for every* y *in* V*, there is a unique sequence of scalars* $c_1, c_2, c_3, \ldots$ *such that*

$$y = c_1 b_1 + c_2 b_2 + c_3 b_3 + \cdots .$$

In a sense, we can say that the elements of the basis "generate" the entire vector space.

Basis

Exercise

Let V be a vector space with bases B and B'. Show that B and B' have the same number of elements.

Basis

Definition

The **dimension** *of a vector space is the cardinality of a bases. If a vector space has a finite basis, then the dimension is written* $dim(V)$.

Basis

Example

Let us go through all the examples given in Example 55 of vector spaces and describe particularly simple bases. Such simple bases are called **canonical bases**. *We also determine the dimension of the vector space.*

- *For* $\mathbf{C}^{m \times n}$, *the canonical basis consists of the mn matrices where each matrix has a single 1 as an entry and all the other entries are 0. In detail,the first three element of the basis look like this*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 0 \end{bmatrix}$$

Linear Maps

The following theorem will have many consequences.

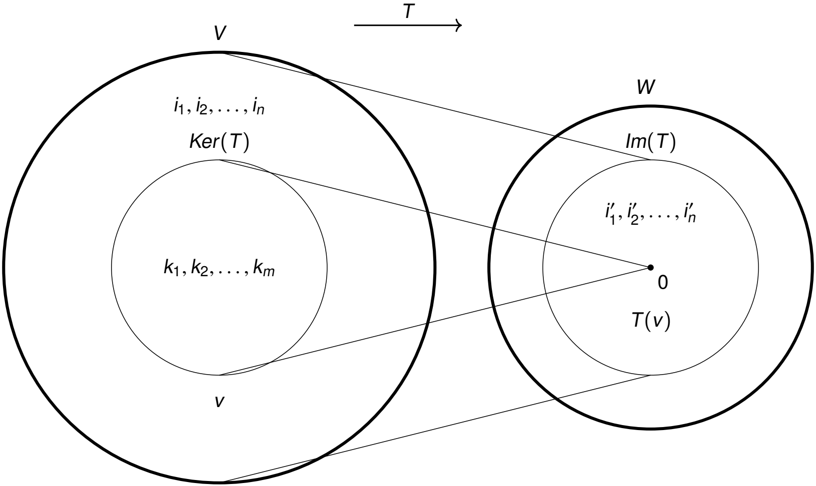
Theorem

Fundamental theorem of linear algebra *or the* **rank-nullity theorem**. *Let V and W be finite dimensional vector spaces. For any linear map $T\colon V \longrightarrow W$, we have the following:*

$$dim(Ker(T)) + dim(Im(T)) = dim(V).$$

Linear Map

This figure helps one "see" the theorem.



Bases for the kernel and image of a linear transformation.

Theorem

Let V and W be finite dimensional vector spaces. The vector spaces V and W are isomorphic if and only if $dim(V) = dim(W)$. This means that for each positive integer m, all the vector spaces of dimension m are isomorphic. We might choose the representative vector space of dimension m as $\mathbf{C}^m$.

There are many ways of describing new vector spaces from existing ones. We have already seen the notion of a subvector space and an inclusion linear map. Let us examine several other methods.

Let W be a vector space with subspaces V and V'. We define the **addition of subspaces** as

$$V + V' = \{v + v' : v \in V \text{ and } v' \in V'\}.$$

To see that this is, in fact, a subspace, notice that all the vectors are elements of W and the addition and the scalar multiplication are both inherited from W. In particular, $V + V'$ is closed under addition. The only thing that remains to be shown is that 0 is in $V + V'$. This can easily be seen because $0 \in V$, $0 \in V'$ and $0 + 0 = 0$.

Let V and V' be vector spaces with $V \cap V' = 0$. If B is a basis of V and B' is a basis of V', then it is not hard to see that $B \cup B'$ is a basis of $V + V'$. And hence, the dimension of $dim(V + V') = dim(V) + dim(V')$.

Function Space

We saw that given sets S and T, the collection of all functions from S to T is a set T^S or $Hom_{\mathbb{Set}}(S, T)$. There is a similar idea for the collection of all linear maps from one vector space to another.

Theorem

For any vector spaces V and W, the collection of all linear maps from V to W, $Hom_{\mathbf{C}\mathbb{Vect}}(V, W)$, is a vector space and is called the **function space***.*

The proof consists of showing that all operations are inherited from W.

Notice that much of the structure of $Hom_{\mathbf{C}\mathbb{V}\mathfrak{ect}}(V, W)$ was inherited from W. This is another instance of Important Categorical Idea 5. When a category has the property that the Hom sets are also objects in the category, then the category is called a **closed category**. We will meet this concept in depth in Chapter **??** and Section **??**.

In terms of dimension, suffice it to say that for finite dimensional vector spaces

$$dim(Hom_{\mathbb{V}\mathfrak{ect}}(V, W)) = dim(V) \times dim(W).$$

Theorem 12 gives us many new examples of vector spaces. In particular, take W to be the complex numbers, $\mathbf{C}$, then $Hom_{\mathbb{V}\mathfrak{ect}}(V, \mathbf{C})$ is a complex vector space. We call this the **dual space** of V and write it as V^*. Remember the objects of V^* are linear maps from V to $\mathbf{C}$. The operations are basically inherited from $\mathbf{C}$. Equation (167) tells us that $dim(V^*) = dim(V) \times 1 = dim(V)$.