Stabilization of Additive Functors

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December 5, 2018
Plan of the talk

Part 1. Homological kindergarten
Part 2. Stabilization of additive functors
Part 3. An extension of the Auslander-Reiten formula
Part 4. Asymptotic stabilization of the tensor product
Part 5. Definition of torsion
Part 6. Definition of cotorsion
Part 7. Duality: the Auslander-Gruson-Jensen functor and friends


Part 1. Homological kindergarten
**Projective resolutions**

**Blanket assumption**: all functors are from modules to abelian groups:

\[ F : \text{Mod}(\Lambda) \rightarrow \text{Ab} \]

and are additive.

Given a module \( M \), an exact sequence

\[ \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \]

(excluding \( M \) itself), where the \( P_i \) are projective, is called a **projective resolution** of \( M \).

**Theorem**

Any two projective resolutions of \( M \) are homotopy equivalent.
Injective resolutions

Given a module $M$, an exact sequence

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \ldots$$

(excluding $M$ itself), where the $I^i$ are injective, is called an injective resolution of $M$.

Theorem

Any two injective resolutions of $M$ are homotopy equivalent.
**Nomenclature: Derived Functors**

Given an additive functor $F$, apply it to a resolution. Since resolutions are homotopically unique, the homology groups of the resulting complex are unique up to isomorphism.

<table>
<thead>
<tr>
<th></th>
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</tr>
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<td><strong>Covariant $F$</strong></td>
<td>$L_i F$</td>
<td>$R^i F$</td>
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<tr>
<td><strong>Contravariant $F$</strong></td>
<td>$R_i F$</td>
<td>$L^i F$</td>
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N.B. For a contravariant $F$, subscripts and superscripts are flipped.
**The Case** \( i = 0: \ L_0F \)

Zeroth derived functors are of special interest to us:

\[
\begin{array}{c}
F(P_1) \rightarrow F(P_0) \rightarrow L_0F(M) \rightarrow 0 \\
\exists! \lambda_F(M) \downarrow \\
F(M)
\end{array}
\]

**Proposition**

\( L_0F \) is right-exact (i.e., preserves cokernels).

**Corollary**

\( \lambda_F : L_0F \rightarrow F \) is an isomorphism if and only if \( F \) is right-exact.
The case $i = 0$: $R^0 F$

\[ \begin{array}{cccc}
0 & \rightarrow & R^0 F(M) & \rightarrow & F(I^0) & \rightarrow & F(I^1)
\end{array} \]

**Proposition**

$R^0 F$ is left-exact (i.e., preserves kernels).

**Corollary**

$\rho_F : F \rightarrow R^0 F$ is an isomorphism if and only if $F$ is left-exact.
### The Case $i = 0$: A Summary

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<td>$L^0 F \xrightarrow{\lambda_F} F$</td>
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- The $\lambda_F$ are isomorphisms if and only if $F$ is right-exact;
- The $\rho_F$ are isomorphisms if and only if $F$ is left-exact.
**Question:** Why is case \( i = 0 \) interesting?

**Answer:** Because all arrows are universal:

- The \( \lambda_F \) – with respect to natural transformations from right-exact functors to \( F \),
- The \( \rho_F \) – with respect natural transformations from \( F \) to left-exact functors.
The universal property of $\lambda$

This diagram shows that the subcategory of right-exact functors is \textbf{coreflective} in the category of all additive functors and

$\lambda$ is a coreflector, i.e., a counit of the adjunction $\iota \dashv L_0$
The universal property of $\rho$

This diagram shows that the subcategory of left-exact functors is **reflective** in the category of all additive functors and

\[ F \xrightarrow{\rho F} R^0 F \]

\[ \forall \xrightarrow{\exists !} \]

\[ Lex \]

**Remark**

$\rho$ is a reflector, i.e., a unit of the adjunction $R^0 \dashv \iota$
An intrinsic characterization of $\lambda$

**Lemma**

If $F$ is covariant (resp., contravariant), then $\lambda_F : L_0 F \rightarrow F$ (resp., $\lambda_F : L^0 F \rightarrow F$) evaluates to an isomorphism on projectives (resp., injectives).

**Proposition**

If $F$ is covariant (resp., contravariant), then $\lambda_F : L_0 F \rightarrow F$ (resp., $\lambda_F : L^0 F \rightarrow F$) is the unique natural transformation from a right-exact functor to $F$ which evaluates to an isomorphism on projectives (resp., injectives).
**An intrinsic characterization of $\rho$**

**Lemma**

If $F$ is covariant (resp., contravariant), then $\rho_F : F \to R^0 F$ (resp., $\rho_F : F \to R_0 F$) evaluates to an isomorphism on injectives (resp., projectives).

**Proposition**

If $F$ is covariant (resp., contravariant), then $\rho_F : F \to R^0 F$ (resp., $\rho_F : F \to R_0 F$) is the unique natural transformation from $F$ to a left-exact functor which evaluates to an isomorphism on injectives (resp., projectives).
Do we only need left- and right-exact functors?

**Remark**
Since $\lambda_F : L_0 F \longrightarrow F$ evaluates to an isomorphism on projectives,

$$L_i \lambda_F : L_i(L_0 F) \longrightarrow L_i F$$

is an isomorphism for all $i$.

**Remark**
Since $\rho_F : F \longrightarrow R^0 F$ evaluates to an isomorphism on injectives,

$$R^i \rho_F : R^i F \longrightarrow R^i (R^0 F)$$

is an isomorphism for all $i$. 
Example: $L_0$ of the covariant Hom

**Example**

$F := (A, -)$, where $A$ is **finitely generated**.

In this case, $L_0 F \xrightarrow{\lambda} F$ is the canonical transformation

$$A^* \otimes - \to (A, -)$$

because it is an isomorphism on projectives and $A^* \otimes -$ is right-exact.

Hence

$$L_i(A, -) \simeq \text{Tor}_i(A^*, -)$$

for all $i$.  

**EXAMPLE:** $R^0$ OF THE TENSOR PRODUCT

**Example**

$$F := A \otimes \_,$$ where $A$ is **finitely presented**.

In this case, $F \xrightarrow{\rho} R^0 F$ is the canonical transformation

$$A \otimes \_ \rightarrow (A^*, \_)$$

because it is an isomorphism on injectives and $(A^*, \_)$ is left-exact.

Hence

$$R^i(A \otimes \_) \cong \text{Ext}^i(A^*, \_)$$

for all $i$. 
Let $F : \text{Mod} (\Lambda) \to \text{Ab}$ be an additive covariant functor. The natural transformation

\[
F(\Lambda) \otimes \_ \xrightarrow{\tau} F
\]

\[
F(\Lambda) \otimes M \xrightarrow{\tau_M} F(M)
\]

\[
x \otimes m \xrightarrow{r_m} F(r_m)(x)
\]

where $x \in F(\Lambda)$, $m \in M$, and $r_m : \Lambda \to M : \lambda \mapsto \lambda m$, evaluates to the canonical isomorphism on $\Lambda$. Whence
Proposition

If \( F \) commutes with coproducts, then \( \tau : F(\wedge) \otimes - \rightarrow F \) evaluates to an isomorphism on projectives, and therefore

\[
F(\wedge) \otimes - \simeq L_0 F
\]

This explains the example with \( L_0(A, _) \).
First applications: recognizing the tensor product

As a consequence,

**Theorem (Eilenberg, Watts)**

If a covariant functor $F$ commutes with coproducts and is right-exact, then $F \simeq F(\wedge) \otimes -$. 
Let $F : \text{Mod}(\Lambda) \to \text{Ab}$ be an additive contravariant functor. The natural transformation

$$F \xrightarrow{\sigma} (\_ , F(\Lambda))$$

$$F(M) \xrightarrow{\sigma_M} (M, F(\Lambda))$$

where $x \in F(M)$, $m \in M$, and $r_m : \Lambda \to M : \lambda \mapsto \lambda m$, evaluates to the canonical isomorphism on $\Lambda$. Whence
Recognizing $R_0F$

**Proposition**

If a contravariant functor $F$ converts coproducts into products, then

$$\sigma : F \longrightarrow (\_ , F(\wedge))$$

evaluates to an isomorphism on projectives, and therefore

$$(\_ , F(\wedge)) \simeq R_0F$$
As a consequence,

**Theorem (Eilenberg, Watts)**

*If a contravariant functor $F$ converts coproducts into products and is left-exact, then $F \simeq (\_, F(\wedge))$.***
Part 2. Stabilization of additive functors
Let $F : \Lambda\text{-Mod} \to \text{Ab}$ be an additive covariant functor on left modules.

**Definition**

The injective stabilization $\overline{F}$ of $F$ is defined by the exact sequence

$$0 \to \overline{F} \to F \xrightarrow{\rho_F} R^0F$$

**Remark**

$\overline{F}$ is additive as a subfunctor of the additive functor $F$. 
Computing the injective stabilization: three easy steps

Let $B$ be a left $\Lambda$-module. To compute $\overline{F}(B)$:

- embed $B$ in an injective: $0 \to B \xrightarrow{i} I$,
- apply $F$
- compute $\text{Ker } F(I)$.

Thus, $\overline{F}(B)$ is defined by the exact sequence

$$0 \to \overline{F}(B) \to F(B) \xrightarrow{F(I)} F(I)$$
Change of notation: The injective stabilization of $F := A \otimes -$ will be denoted by $A \overline{\otimes} -$ . Thus

$$A \overline{\otimes} B = (A \overline{\otimes} -)(B)$$

Terminology: $A$ is inert, $B$ is active.

**Example**

Take $\Lambda := \mathbb{Z}$. Then:

- $\mathbb{Z} \overline{\otimes} \mathbb{Q}/\mathbb{Z} = 0$ as $\mathbb{Q}/\mathbb{Z}$ is injective (or $\mathbb{Z}$ is projective);
- $\mathbb{Z} \overline{\otimes} \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$ (just tensor $0 \to \mathbb{Z} \to \mathbb{Q}$ with $\mathbb{Q}/\mathbb{Z}$).

N.B. The harpoon always points to the active variable.
Example

If $A$ is finitely presented, then (A-B, 1969)

$$A \tilde{\otimes} \_ \cong \text{Ext}^1(\text{Tr}A, \_ )$$

Definition (for later use)

$$s(A) := A \tilde{\otimes} \Lambda \Lambda$$
Projective stabilization of $\text{Hom}(A, \_)$

Similar definitions apply to the remaining three choices for $\lambda$ and $\rho$.

**Example**

Let $A$ be a left $\Lambda$-module. The projective stabilization of $(A, \_)$ is just $(\underline{A}, \_)$, the Hom modulo projectives.

If $A$ is finitely presented, then

$$(A, \_) \simeq \text{Tor}_1(\text{Tr} A, \_)$$
**Example**

Let $B$ be a left $\Lambda$-module. The projective stabilization of $(\_, B)$ is just $(\_, B)$, the Hom modulo injectives (sic!).

**Definition (for later use)**

$$q(B) := (\Lambda, B)$$
**Stabilization: Summary**

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<thead>
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<td>$\mathbb{1} \rightarrow F \xrightarrow{\rho} R^0F$</td>
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<td>$L^0F \xrightarrow{\lambda_F} F \rightarrow \bigoplus$</td>
</tr>
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</table>

- Projective stabilization $= \text{the cokernel of the counit } \lambda$.
- Injective stabilization $= \text{the kernel of the unit } \rho$. 
Part 3. An extension of the Auslander-Reiten formula
Let $\Lambda$ be an algebra over a commutative ring $R$ (can be $\mathbb{Z}$). Choose an injective $R$-module $J$ and let $D_J := \text{Hom}_R(_{-}, J)$.

**Proposition**

The tensor – covariant Hom adjunction induces an isomorphism

$$D_J(A \widehat{\otimes} B) \cong \overline{\text{Hom}}(B, D_J(A)),$$

functorial in $A$ and $B$. 
The original Auslander-Reiten formula

**Remark**

If $A$ in $D_{\mathcal{J}}(A \otimes B) \simeq \text{Hom}(B, D_{\mathcal{J}}(A))$ is finitely presented, then:

- $A$ is projectively equivalent to $\text{Tr } A'$ for some $A'$,
- $\text{Tr } A' \otimes B$ is well-defined (tensoring by a projective is exact),
- $D_{\mathcal{J}}(\text{Tr } A')$ is determined uniquely modulo injectives,
- $A \otimes _{-} \simeq \text{Ext}^1(\text{Tr } A, _{-}) \simeq \text{Ext}^1(A', _{-})$.

This yields an isomorphism

$$D_{\mathcal{J}} \text{Ext}^1(A', B) \simeq \text{Hom}(B, D_{\mathcal{J}} \text{Tr } A')$$

which is the original Auslander-Reiten formula.
SPECIALIZING TO $B := \Lambda$

Setting $B := \Lambda$, we have

$$D_J(A \otimes \Lambda) \cong \text{Hom}(\Lambda, D_J(A))$$

or, using earlier notation:

$$D_J(\mathfrak{g}(A)) \cong q(D_J(A))$$
Part 4. Asymptotic stabilization of the tensor product
Q: Is there a **homological** counterpart of Buchweitz’s construction?
Let $\Lambda$ be a ring. Given $\Lambda$-modules $M$ and $N$, we have a sequence of maps of abelian groups

$$(M, N) \rightarrow (\Omega M, \Omega N) \rightarrow (\Omega^2 M, \Omega^2 N) \rightarrow \ldots$$

Definition

$$B^0(M, N) := \lim_{k \geq 0} (\Omega^k M, \Omega^k N)$$

Key point: each term in the defining sequence is a value of the projective stabilization of the covariant Hom functor: $(\Omega^k M, \_)(\Omega^k N)$. 
### Dualizing Buchweitz’s construction

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Covariant Hom</td>
<td>$\otimes$</td>
</tr>
<tr>
<td>Projective stabilization</td>
<td>Injective stabilization</td>
</tr>
<tr>
<td>$\Omega$ on the active (co) side</td>
<td>$\Sigma$ on the active side</td>
</tr>
<tr>
<td>$\Omega$ on the inert (contra) side</td>
<td>$\Omega$ on the inert side</td>
</tr>
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</table>

As a result, we have a sequence of abelian groups

\[
\ldots \Omega^2 A \otimes \Sigma^2 B \quad \Omega A \otimes \Sigma B \quad A \otimes B
\]

but, so far, no maps.
To define maps, tensor a syzygy sequence for $A$ with a cosyzygy sequence for $B$. The connecting homomorphism from the snake lemma induces a map $\Omega A \overset{\cong}{\otimes} \Sigma B \longrightarrow A \overset{\cong}{\otimes} B$.

Iteration yields the desired sequence of maps:

$$\ldots \longrightarrow \Omega^2 A \overset{\cong}{\otimes} \Sigma^2 B \longrightarrow \Omega A \overset{\cong}{\otimes} \Sigma B \longrightarrow A \overset{\cong}{\otimes} B$$
The asymptotic stabilization $T_n(A, -)$ of the left tensor product in degree $n$ with coefficients in the right $\Lambda$-module $A$ is defined by

$$T_n(A, -)(B) := T_n(A, B) := \lim_{k, k+n \geq 0} \Omega^{k+n} A \otimes \Sigma^k B$$

This produces a homological counterpart to Buchweitz’s construction.
COMPARING $V$ AND $T$

**Theorem**

Let $A$ be a right $\Lambda$-module. For each $l \in \mathbb{Z}$, there is an epimorphic natural transformation

$$\kappa_l : V_l(A, \_ ) \to T_l(A, \_ )$$
Theorem

For any module $A$, there is a commutative diagram of connected sequences of functors

$$
\begin{array}{c}
V_\bullet(A, -) \\
\kappa \quad \theta \\
T_\bullet(A, -) \quad \cong \quad M_\bullet(\operatorname{Tor}(A, -)) \\
\lambda \quad \tau \\
\operatorname{Tor}(A, -)
\end{array}
$$

where the horizontal arrow is an isomorphism.
**What is $\text{Ker} \kappa$? A conjecture**

The comparison map $\kappa : V_\bullet(A, _) \longrightarrow T_\bullet(A, _) \,$ appears to be an algebraic analog of the comparison map from Steenrod-Sitnikov homology to Čech homology. That map is also epic, and its kernel is given by the first derived limit. Based on this analogy, we had

**Conjecture (2014)**

$\text{Ker} \kappa$ is given by the first derived limit.
A positive answer

The conjecture is true by the following recent result.

**Theorem (I. Emmanouil and P. Manousaki, 2017)**

There is an exact sequence

\[
0 \longrightarrow \lim_{\leftarrow i} \text{Tor}_{i}^{+1} \left( A, \Sigma^i \_ \right) \longrightarrow V_\bullet \left( A, \_ \right) \longrightarrow M_\bullet \left( \text{Tor} \left( A, \_ \right) \right) \longrightarrow 0
\]
Part 5. Definition of torsion
Definition of torsion

What are we trying to define?

- For any module over any ring, define the torsion submodule, extending classical torsion over commutative domains.
- For any module over any ring, define the cotorsion quotient module.
**Classical torsion**

Classical torsion is defined over commutative domains:

\[ T(A) := \{ a \in A \mid \exists r \in R - \{0\}, ra = 0 \} \]

It can be extended to arbitrary rings in more than one way, but we want to simultaneously generalize the notion of 1-torsion.
**What is 1-torsion?**

The 1-torsion $t(A)$ of a module $A$ is the kernel of the canonical evaluation map

$$e_A : A \rightarrow A^{**} : a \mapsto (F_a : f \mapsto f(a))$$

Thus $t(A)$ is determined by the exact sequence

$$0 \rightarrow t(A) \rightarrow A \rightarrow A^{**}$$

It is defined for any module over any ring and consists of those elements of $A$ on which every linear form on $A$ vanishes. Moreover:

**Lemma**

*If $R$ is a commutative domain and $A$ is finitely generated, then*

$$T(A) = t(A)$$
**Warning: 1-torsion can be big**

For infinite modules, 1-torsion need not coincide with classical torsion.

**Example**

Let $R := \mathbb{Z}$ and $A := \mathbb{Q}$. Then

\[ T(\mathbb{Q}) = \{0\} \quad \text{but} \quad t(\mathbb{Q}) = \mathbb{Q} \]
However, 1-torsion is a ubiquitous concept:

- Stable module theory (Auslander - Bridger, 1969);
- PDE and constructive aspects of linear control systems (Oberst, et al. 1990, ...);
- Linkage of algebraic varieties (M - Strooker, 2004);
- Algebraic aspects of a question of Reiffen - Vetter (M, 2010);
- Local algebraic geometry, singularity theory, local cohomology, ...
**Problem** Find a common generalization of:

- classical torsion for arbitrary modules over commutative domains, and
- 1-torsion for finitely presented modules over arbitrary rings.

The new definitions should work for arbitrary modules over arbitrary rings.
Definition of torsion

**Classical torsion via localization**

Let $K$ be the field of fractions of the commutative domain $R$ and

\[ 0 \longrightarrow R \longrightarrow K \]

the canonical embedding.

Tensoring a module $A$ with this map, we have the localization map

\[ \ell_A : A \cong A \otimes R \rightarrow A \otimes_R K \]

**Lemma**

\[ T(A) = \text{Ker} \, \ell_A. \]

**Observation**: In this construction, $K$ is the injective envelope of the ring.
Let $\Lambda$ be a ring, $A$ a right $\Lambda$-module, and

$$0 \longrightarrow \Lambda \longrightarrow I$$

the injective envelope of $\Lambda$ viewed as a left module over itself.

**Definition**

The torsion of $A$ is defined by the exact sequence

$$0 \longrightarrow \mathfrak{s}(A) \longrightarrow A \otimes \Lambda \longrightarrow A \otimes I$$
PROPOSITION

- If $\Lambda$ is a commutative domain, then $s$ coincides with classical torsion.

- On finitely presented modules over any $\Lambda$, $s$ coincides with 1-torsion.
**First properties of $\mathfrak{s}$**

**Theorem**

- $\mathfrak{s}$ is a subfunctor of 1-torsion: $\mathfrak{s} \subseteq t$.
- $\mathfrak{s}$ preserves filtered colimits (and hence coproducts).
- $\mathfrak{s}$ is the largest subfunctor of $t$ that preserves filtered colimits.
- $\mathfrak{s}$ is a radical, i.e., $\mathfrak{s}(A/\mathfrak{s}(A)) = \{0\}$ for any module $A$.
- $\mathfrak{s}(A) = 0$ for any flat module $A$. 
FURTHER PROPERTIES OF $\mathfrak{s}$

**Proposition**

The following conditions are equivalent:

A) $\mathfrak{s}$ is the zero functor (on right modules);

B) $\mathfrak{s}$ preserves epimorphisms;

C) $\Lambda$ is absolutely pure;

D) $\Lambda$ is left FP-injective, i.e., $\text{Ext}_\Lambda^1(M, \Lambda) = \{0\}$ for all finitely presented left $\Lambda$-modules $M$.

In particular, if $\Lambda$ is selfinjective on the left, then $\mathfrak{s}$ is the zero functor.
Let $\text{Rej}(A, \mathcal{F})$ be the reject of the class $\mathcal{F}$ of flats in the right module $A$, and $\text{rej}(A, \mathcal{F})$ its restriction to finitely presented modules. Then

$$s \subseteq \text{Rej}(\_ , \mathcal{F}) \subseteq t$$

Restricting to finitely presented modules, we get equalities. Whence

**Proposition**

$s \cong \overset{\rightarrow}{\text{rej}}(\_ , \mathcal{F})$, i.e., the torsion functor is isomorphic to the colimit extension of the reject of flats restricted to finitely presented modules.
Most of the basic results about classical torsion carry over, in one form or another, to the new setting. State such results and prove them.
Part 6. Definition of cotorsion
How do we define cotorsion?

In the absence of a classical prototype, we try and dualize the definition of torsion.

Start with a simple question:

- Why is $s(A)$ a subset of $A$?
WHY IS $\mathfrak{s}(A)$ A SUBSET OF $A$

The answer is obvious: because, by definition, $\mathfrak{s}(A)$ is a subset of $A \otimes \Lambda$ and there is a canonical isomorphism

$$A \otimes \Lambda \overset{\cong}{\longrightarrow} A$$

Question Is there a “dual” canonical isomorphism?

Answer Yes, there is:

$$\text{Hom}(\Lambda, C) \overset{\cong}{\longrightarrow} C$$
The torsion of \( A \) was defined as the value of the \textit{injective} stabilization of the tensor product functor \( A \otimes \_ \) on \( \Lambda \).

Dually, we define the cotorsion of \( C \) as the value of the \textit{projective} stabilization of the contravariant Hom functor \( \text{Hom}(\_, C) \) on \( \Lambda \).
Let $C$ be a (left) $\Lambda$-module. The cotorsion quotient module of $C$ is

$$q(C) := \underline{\text{Hom}}(\_ , C)(\Lambda) = \underline{\text{Hom}}(\Lambda , C)$$

Thus $q = \underline{\text{Hom}}(\Lambda , \_)$ is a quotient of the identity functor.
**FIRST OBSERVATIONS**

- The short exact sequences

\[ 0 \rightarrow I(\Lambda, C) \rightarrow (\Lambda, C) \rightarrow (\overline{\Lambda}, C) \rightarrow 0 \]

give rise to a short exact sequence of endofunctors on \( \Lambda\text{-Mod} \)

\[ 0 \rightarrow q^{-1} \rightarrow 1 \rightarrow q \rightarrow 0 \]

- \( q \) preserves epimorphisms.

- \( q \) is finitely presented:

\[ (I, \_ ) \xrightarrow{(\iota, \_ )} (\Lambda, \_ ) \rightarrow q \rightarrow 0 \]
**Lemma**

Under the canonical isomorphism

\[(\Lambda, C) \cong C : f \mapsto f(1),\]

\(I(\Lambda, C)\) identifies with \(Tr(I, C)\), the trace in \(C\) of the class \(I\) of injective \(\Lambda\)-modules.

**Proposition**

\(q\) is a coradical, i.e., \(q(q^{-1}(C)) = \{0\}\) for any \(C\).
**Cotorsion Modules**

**Definition**

The module $C$ is cotorsion if $C \rightarrow q(C)$ is an isomorphism. In other words, $C$ is cotorsion if no map $\Lambda \rightarrow C$ factors through an injective. Equivalently, $Tr(J, C) = \{0\}$.

**Example**

Any PID which is not a field, viewed as a module over itself, is cotorsion (as it has no nonzero divisible elements).
**Cotorsion-free modules**

**Definition**

The module $C$ is cotorsion-free if $C \to q(C)$ is the zero map, i.e., any map $\Lambda \to C$ factors through an injective. Equivalently, $Tr(\mathcal{I}, C) = C$.

**Example**

Any injective module is cotorsion-free.

Obviously, $\{0\}$ is the only module which is cotorsion and cotorsion-free.
EXPECTED PROPERTIES HOLD

EXERCISE
Formulate and prove basic properties of cotorsion (Hint: dualize the properties of torsion).
Part 7. The Auslander-Gruson-Jensen functor and friends
The Auslander-Gruson-Jensen duality, discovered by Auslander and independently by Gruson and Jensen, is a pair of exact contravariant functors

\[ \text{fp}(\text{mod}(\Lambda^{op}), \text{Ab}) \xrightarrow{D} \text{fp}(\text{mod}(\Lambda), \text{Ab}) \]

each of which interchanges the tensor product and the Hom functor when the fixed argument is a finitely presented module.
AN EXTENSION OF THE AGJ FUNCTOR

There is an exact contravariant functor

\[ D_A : fp(\text{Mod} (\Lambda^{op}), \text{Ab}) \to (\text{mod}(\Lambda), \text{Ab}) \]

defined by

\[ D_A := R_0(\epsilon \circ w), \]

where \( \epsilon \) is the tensor embedding

\[ \epsilon : \text{Mod} (\Lambda^{op}) \to (\text{mod}(\Lambda), \text{Ab}) : M \mapsto _- \otimes M \]

and \( w \) is the defect functor. For any representable functor \((M, _-)

\[ D_A(M, _) = _- \otimes M \]

As is shown by Dean-Russell (2016), the functor \( D_A \) is completely
determined by this property and by being exact.
Theorem (S. Dean - J. Russell, 2016)

The functor

\[ D_A : \text{fp}(\text{Mod}(\Lambda^{\text{op}}), Ab) \to (\text{mod}(\Lambda), Ab) \]

admits a left adjoint \( D_L \) and a right adjoint \( D_R \), both of which are fully faithful. The functors \( D_R \) and \( D_A \) restrict to the AGJ duality \( D \) on the full subcategories of pp-functors.
The foregoing statement is part of the following diagram of functors

\[
\begin{array}{c}
\text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab}) \\
\text{Mod}(\Lambda^{op}) \\
(\text{mod}(\Lambda), \text{Ab})
\end{array}
\]

\[
\begin{array}{c}
D_L \\
D_A \\
D_R
\end{array}
\]

\[
\begin{array}{c}
w \\
L^0Y \\
Y \\
R_0\epsilon \\
\epsilon \\
\text{ev}_\Lambda
\end{array}
\]
**Theorem**

For any module $B$

$$DA\text{Hom}(B, \_ ) \simeq \_ \hat{\otimes} B$$

**Corollary**

Specializing to $B := \wedge$, we have

$$DA(q) \simeq s$$

(dexterity changes). Equivalently,

$$DA(Tr(J, \_ )^{-1}) \simeq \text{rej}(\_, \mathcal{F})$$
PROPOSITION

For any pure injective left $\Lambda$-module $M$,

$$\text{Hom}(M, _) \cong D_L(\_ \otimes M)$$

COROLLARY

If $\Lambda$ is pure injective, then $q \cong D_L(s)$.
**Theorem**

Suppose the injective envelope of $\Lambda$ is finitely presented. Then the notions of torsion and cotorsion are dual. More precisely, the right adjoint

$$D_R : (\text{mod}(\Lambda), Ab) \to \text{fp}(\text{Mod}(\Lambda^{\text{op}}), Ab)$$

of $D_A$ carries the torsion functor to the cotorsion functor, i.e.,

$$D_R(s) \cong q$$

**Corollary**

Let $\Lambda$ be an artin algebra. Then $D_R(s) \cong q$. 
The Auslander-Reiten formula for arbitrary modules, again

The foregoing isomorphisms are between functors, with no apparent connections between their arguments. We can do better (slide 36):

\[ D_J(\mathfrak{s}(A)) \cong q(D_J(A)) \]

Thus, the dual of the torsion of a module is the cotorsion of the dual of the module.
**Example**

Let $\Lambda$ be any ring, $R := \mathbb{Z}$, and $J := \mathbb{Q}/\mathbb{Z}$. Set $(\_)^+ := DJ(\_).$ Then

$$\delta(A)^+ \simeq q(A^+)$$