Stabilization of Additive Functors

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Plan of the talk

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Part 5. Definition of torsion

Part 6. Definition of cotorsion

Part 7. Duality: the Auslander-Gruson-Jensen functor and friends

References

A. Martsinkovsky and J. Russell, *Injective stabilization of additive functors. I. Preliminaries*, arXiv:1701.00150, 2017

A. Martsinkovsky and J. Russell, *Injective stabilization of additive functors. II. (Co)torsion and the Auslander-Gruson-Jensen functor*, arXiv:1701.00151, 2017

A. Martsinkovsky and J. Russell, *Injective stabilization of additive functors. III. Asymptotic stabilization of the tensor product*, arXiv:1701.00268, 2017

Part 1. Homological kindergarten

Projective resolutions

Blanket assumption: all functors are from modules to abelian groups:

$$F : \mathrm{Mod}(\Lambda) \longrightarrow \mathtt{Ab}$$

and are additive.

Given a module M, an exact sequence

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

(excluding M itself), where the P_i are projective, is called a **projective resolution** of M.

Theorem

Any two projective resolutions of M are homotopy equivalent.

Injective resolutions

Given a module M, an exact sequence

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

(excluding M itself), where the I^i are injective, is called an **injective resolution** of M.

Theorem

Any two injective resolutions of M are homotopy equivalent.

NOMENCLATURE: DERIVED FUNCTORS

Given an additive functor F, apply it to a resolution. Since resolutions are homotopically unique, the homology groups of the resulting complex are unique up to isomorphism.

	Projective resolutions	**Injective resolutions**
Covariant F	$L_i F$	$R^i F$
Contravariant F	$R_i F$	$L^i F$

N.B. For a contravariant F, subscripts and superscripts are flipped.

THE CASE $i = 0$: L_0F

Zeroth derived functors are of special interest to us:

$$\begin{tikzcd}
F(P_1) \arrow[r] & F(P_0) \arrow[r] \arrow[dr] & L_0F(M) \arrow[r] \arrow[d, dotted, "\exists!\, \lambda_F(M)"] & 0 \\
& & F(M) &
\end{tikzcd}$$

PROPOSITION

L_0F is right-exact (i.e., preserves cokernels).

COROLLARY

$\lambda_F : L_0F \longrightarrow F$ is an isomorphism if and only if F is right-exact.

THE CASE $i = 0$: R^0F

$$
\begin{tikzcd}
 & F(M) \arrow[d, dotted, "\exists!\ \rho_F(M)"'] \arrow[dr] & & \\
0 \arrow[r] & R^0F(M) \arrow[r] & F(I^0) \arrow[r] & F(I^1)
\end{tikzcd}
$$

PROPOSITION

R^0F is left-exact (i.e., preserves kernels).

COROLLARY

$\rho_F : F \longrightarrow R^0F$ is an isomorphism if and only if F is left-exact.

The case $i = 0$: a summary

	Projective resolutions	**Injective resolutions**
Covariant F	$L_0F \xrightarrow{\lambda_F} F$	$F \xrightarrow{\rho_F} R^0F$
Contravariant F	$F \xrightarrow{\rho_F} R_0F$	$L^0F \xrightarrow{\lambda_F} F$

- The λ_F are isomorphisms if and only if F is right-exact;
- The ρ_F are isomorphisms if and only if F is left-exact.

Question: why is case $i = 0$ interesting?

	Projective resolutions	**Injective resolutions**
Covariant F	$L_0F \xrightarrow{\lambda_F} F$	$F \xrightarrow{\rho_F} R^0F$
Contravariant F	$F \xrightarrow{\rho_F} R_0F$	$L^0F \xrightarrow{\lambda_F} F$

Answer: because all arrows are universal:

- the λ_F – with respect to natural transformations from right-exact functors to F,
- the ρ_F – with respect natural transformations from F to left-exact functors.

The universal property of λ

$$\begin{tikzcd}
 & Rex \arrow[dl, dashed, "\exists\,!"'] \arrow[d, "\forall"] \\
L_0F \arrow[r, "\lambda_F"] & F
\end{tikzcd}$$

Remark

This diagram shows that the subcategory of right-exact functors is **coreflective** in the category of all additive functors and

λ is a coreflector, i.e., a counit of the adjunction $\iota \dashv L_0$

The universal property of ρ

$$\begin{tikzcd} F \arrow[r, "\rho_F"] \arrow[d, "\forall"'] & R^0 F \arrow[dl, dashed, "\exists\,!"] \\ Lex & \end{tikzcd}$$

Remark

This diagram shows that the subcategory of left-exact functors is **reflective** in the category of all additive functors and

ρ is a reflector, i.e., a unit of the adjunction $R^0 \dashv \iota$

AN INTRINSIC CHARACTERIZATION OF λ

LEMMA

If F is covariant (resp., contravariant), then $\lambda_F : L_0F \longrightarrow F$ (resp., $\lambda_F : L^0F \longrightarrow F$) evaluates to an isomorphism on projectives (resp., injectives).

PROPOSITION

If F is covariant (resp., contravariant), then $\lambda_F : L_0F \longrightarrow F$ (resp., $\lambda_F : L^0F \longrightarrow F$) is the ***unique*** *natural transformation from a right-exact functor to F which evaluates to an isomorphism on projectives (resp., injectives).*

AN INTRINSIC CHARACTERIZATION OF ρ

LEMMA

If F is covariant (resp., contravariant), then $\rho_F : F \longrightarrow R^0 F$ (resp., $\rho_F : F \longrightarrow R_0 F$) evaluates to an isomorphism on injectives (resp., projectives).

PROPOSITION

If F is covariant (resp., contravariant), then $\rho_F : F \longrightarrow R^0 F$ (resp., $\rho_F : F \longrightarrow R_0 F$) is the ***unique*** *natural transformation from F to a left-exact functor which evaluates to an isomorphism on injectives (resp., projectives).*

Do we only need left- and right-exact functors?

Remark

Since $\lambda_F : L_0 F \longrightarrow F$ evaluates to an isomorphism on projectives,

$$L_i \lambda_F : L_i(L_0 F) \longrightarrow L_i F$$

is an isomorphism for all i.

Remark

Since $\rho_F : F \longrightarrow R^0 F$ evaluates to an isomorphism on injectives,

$$R^i \rho_F : R^i F \longrightarrow R^i(R^0 F)$$

is an isomorphism for all i.

Example: L_0 of the covariant Hom

Example

$\boxed{F := (A, \_), \text{ where } A \text{ is } \textbf{finitely generated.}}$

In this case, $L_0 F \xrightarrow{\lambda} F$ is the canonical transformation

$$A^* \otimes \_ \longrightarrow (A, \_)$$

because it is an isomorphism on projectives and $A^* \otimes \_$ is right-exact.

Hence

$$L_i(A, \_) \simeq \mathrm{Tor}_i(A^*, \_)$$

for all i.

Example: R^0 of the tensor product

Example

$\boxed{F := A \otimes \_\,\text{, where } A \text{ is } \textbf{finitely presented.}}$

In this case, $F \xrightarrow{\rho} R^0F$ is the canonical transformation

$$A \otimes \_ \longrightarrow (A^*, \_)$$

because it is an isomorphism on injectives and $(A^*, \_)$ is left-exact.

Hence

$$R^i(A \otimes \_) \simeq \mathrm{Ext}^i(A^*, \_)$$

for all i.

RECOGNIZING $L_0 F$

Let $F : \mathrm{Mod}\,(\Lambda) \longrightarrow \mathtt{Ab}$ be an additive **covariant** functor. The natural transformation

$$F(\Lambda) \otimes \_ \xrightarrow{\ \tau\ } F$$

$$F(\Lambda) \otimes M \xrightarrow{\ \tau_M\ } F(M)$$

$$x \otimes m \longmapsto F(r_m)(x)$$

where $x \in F(\Lambda)$, $m \in M$, and $r_m : \Lambda \longrightarrow M : \lambda \mapsto \lambda m$, evaluates to the canonical isomorphism on Λ. Whence

Recognizing L_0F

Proposition

If F commutes with coproducts, then $\tau : F(\Lambda)\otimes \_ \longrightarrow F$ evaluates to an isomorphism on projectives, and therefore

$$F(\Lambda)\otimes \_ \simeq L_0F$$

This explains the example with $L_0(A, \_)$.

First applications: recognizing the tensor product

As a consequence,

Theorem (Eilenberg, Watts)

If a covariant functor F commutes with coproducts and is right-exact, then $F \simeq F(\Lambda) \otimes \_$.

Recognizing R_0F

Let $F : \mathrm{Mod}\,(\Lambda) \longrightarrow \mathtt{Ab}$ be an additive **contravariant functor.** The natural transformation

$$F \xrightarrow{\ \sigma\ } (\_, F(\Lambda))$$

$$F(M) \xrightarrow{\ \sigma_M\ } (M, F(\Lambda))$$

$$x \longmapsto \left(m \longmapsto F(r_m)(x) \right)$$

where $x \in F(M)$, $m \in M$, and $r_m : \Lambda \to M : \lambda \mapsto \lambda m$, evaluates to the canonical isomorphism on Λ. Whence

RECOGNIZING R_0F

PROPOSITION

If a contravariant functor F converts coproducts into products, then $\sigma : F \longrightarrow (\_, F(\Lambda))$ evaluates to an isomorphism on projectives, and therefore

$$(\_, F(\Lambda)) \simeq R_0F$$

First applications: recognizing the contravariant Hom

As a consequence,

Theorem (Eilenberg, Watts)

If a contravariant functor F converts coproducts into products and is left-exact, then $F \simeq (\_, F(\Lambda))$.

Part 2. Stabilization of additive functors

Injective stabilization of an additive functor

Let $F : \Lambda\text{-Mod} \to \mathtt{Ab}$ be an additive **covariant** functor on left modules.

Definition

The injective stabilization $\overline{F}$ of F is defined by the exact sequence

$$0 \longrightarrow \overline{F} \longrightarrow F \xrightarrow{\rho_F} R^0 F$$

Remark

$\overline{F}$ is additive as a subfunctor of the additive functor F.

Computing the injective stabilization: three easy steps

Let B be a left Λ-module. To compute $\overline{F}(B)$:

- embed B in an injective: $0 \to B \xrightarrow{\iota} I$,
- apply F
- compute $\operatorname{Ker} F(\iota)$.

Thus, $\overline{F}(B)$ is defined by the exact sequence

$$0 \longrightarrow \overline{F}(B) \longrightarrow F(B) \xrightarrow{F(\iota)} F(I)$$

INJECTIVE STABILIZATION OF THE TENSOR PRODUCT

Change of notation: The injective stabilization of $F := A \otimes \_$ will be denoted by $A \overrightarrow{\otimes} \_$. Thus

$$A \overrightarrow{\otimes} B = (A \overrightarrow{\otimes} \_)(B)$$

Terminology: A is *inert*, B is *active*.

EXAMPLE

Take $\Lambda := \mathbb{Z}$. Then:

- $\mathbb{Z} \overrightarrow{\otimes} \mathbb{Q}/\mathbb{Z} = 0$ as $\mathbb{Q}/\mathbb{Z}$ is injective (or $\mathbb{Z}$ is projective);
- $\mathbb{Z} \overleftarrow{\otimes} \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$ (just tensor $0 \to \mathbb{Z} \to \mathbb{Q}$ with $\mathbb{Q}/\mathbb{Z}$).

N.B. The harpoon always points to the active variable.

Injective stabilization of the tensor product

Example

If A is finitely presented, then (A-B, 1969)

$$A \overrightarrow{\otimes} \_ \simeq \mathrm{Ext}^1(\mathrm{Tr} A, \_)$$

Definition (for later use)

$$\mathfrak{s}(A) := A \overrightarrow{\otimes}_\Lambda \Lambda$$

PROJECTIVE STABILIZATION OF HOM(A, _)

Similar definitions apply to the remaining three choices for λ and ρ.

EXAMPLE

Let A be a left Λ-module. The projective stabilization of $(A, \_)$ is just $(\underline{A, \_})$, the Hom modulo projectives.

If A is finitely presented, then

$$(\underline{A, \_}) \simeq \mathrm{Tor}_1(\mathrm{Tr}\, A, \_)$$

Projective stabilization of Hom(_ , B)

Example

Let B be a left Λ-module. The projective stabilization of $(\_, B)$ is just $\overline{(\_, B)}$, the Hom modulo injectives (sic!).

Definition (for later use)

$$\mathfrak{q}(B) := (\overline{\Lambda, B})$$

Stabilization: summary

	Projective resolutions	**Injective resolutions**
Covariant	$L_0F \xrightarrow{\lambda_F} F \longrightarrow ⓟ$	$ⓘ \longrightarrow F \xrightarrow{\rho_F} R^0F$
Contravariant	$ⓘ \longrightarrow F \xrightarrow{\rho_F} R_0F$	$L^0F \xrightarrow{\lambda_F} F \longrightarrow ⓟ$

- Projective stabilization = the cokernel of the counit λ.
- Injective stabilization = the kernel of the unit ρ.

Part 3. An extension of the Auslander-Reiten formula

AN AUSLANDER-REITEN FORMULA FOR ARBITRARY MODULES

Let Λ be an algebra over a commutative ring R (can be $\mathbb{Z}$). Choose an injective R-module $\mathbf{J}$ and let $D_{\mathbf{J}} := \mathrm{Hom}_R(\_, \mathbf{J})$.

PROPOSITION

The tensor – covariant Hom adjunction induces an isomorphism

$$D_{\mathbf{J}}(A \overrightarrow{\otimes} B) \simeq \overline{\mathrm{Hom}}(B, D_{\mathbf{J}}(A)),$$

functorial in A and B.

THE ORIGINAL AUSLANDER-REITEN FORMULA

REMARK

If A in $D_{\mathbf{J}}(A \overrightarrow{\otimes} B) \simeq \overline{\mathrm{Hom}}(B, D_{\mathbf{J}}(A))$ is finitely presented, then:

- A is projectively equivalent to $\mathrm{Tr}\, A'$ for some A',
- $\mathrm{Tr}\, A' \overrightarrow{\otimes} B$ is well-defined (tensoring by a projective is exact),
- $D_{\mathbf{J}}(\mathrm{Tr}\, A')$ is determined uniquely modulo injectives,
- $A \overrightarrow{\otimes} \_ \simeq \mathrm{Ext}^1(\mathrm{Tr}\, A, \_) \simeq \mathrm{Ext}^1(A', \_)$.

This yields an isomorphism

$$D_{\mathbf{J}}\, \mathrm{Ext}^1(A', B) \simeq \overline{\mathrm{Hom}}(B, D_{\mathbf{J}}\, \mathrm{Tr}\, A')$$

which is the original Auslander-Reiten formula.

Specializing to $B := \Lambda$

Setting $B := \Lambda$, we have

$$D_{\mathbf{J}}(A \overrightarrow{\otimes} \Lambda) \simeq \overline{\mathrm{Hom}}(\Lambda, D_{\mathbf{J}}(A))$$

or, using earlier notation :

$$D_{\mathbf{J}}(\mathfrak{s}(A)) \simeq \mathfrak{q}(D_{\mathbf{J}}(A))$$

Part 4. Asymptotic stabilization of the tensor product

TATE (CO)HOMOLOGY WITHOUT COMPLETE RESOLUTIONS

Cohomology	Homology
$\mathrm{V}^i(M, N)$	$\mathrm{V}_i(M, N)$
$\mathrm{B}^i(M, N)$	?
$\mathrm{M}^i F$	$\mathrm{M}_i F$

Q: Is there a **homological** counterpart of Buchweitz's construction?

BUCHWEITZ'S CONSTRUCTION

Let Λ be a ring. Given Λ-modules M and N, we have a sequence of maps of abelian groups

$$(\underline{M, N}) \longrightarrow (\underline{\Omega M, \Omega N}) \longrightarrow (\underline{\Omega^2 M, \Omega^2 N}) \longrightarrow \dots$$

DEFINITION

$$\mathrm{B}^0(M, N) := \varinjlim_{k \geqslant 0} (\underline{\Omega^k M, \Omega^k N})$$

Key point: each term in the defining sequence is a value of the projective stabilization of the covariant Hom functor: $(\underline{\Omega^k M, \_\,})(\Omega^k N)$.

DUALIZING BUCHWEITZ'S CONSTRUCTION

Stable cohomology	**Stable homology**
Covariant Hom	$\otimes$
Projective stabilization	Injective stabilization
Ω on the active (co) side	Σ on the active side
Ω on the inert (contra) side	Ω on the inert side

As a result, we have a sequence of abelian groups

$$\dots \qquad \Omega^2 A \overrightarrow{\otimes} \Sigma^2 B \qquad \Omega A \overrightarrow{\otimes} \Sigma B \qquad A \overrightarrow{\otimes} B$$

but, so far, no maps.

Building maps

To define maps, tensor a syzygy sequence for A with a cosyzygy sequence for B. The connecting homomorphism from the snake lemma induces a map $\Omega A \overset{\rightarrow}{\otimes} \Sigma B \longrightarrow A \overset{\rightarrow}{\otimes} B$.

Iteration yields the desired sequence of maps:

$$\ldots \longrightarrow \Omega^2 A \overset{\rightarrow}{\otimes} \Sigma^2 B \longrightarrow \Omega A \overset{\rightarrow}{\otimes} \Sigma B \longrightarrow A \overset{\rightarrow}{\otimes} B$$

Asymptotic stabilization of the tensor product

Definition

The asymptotic stabilization $\mathrm{T}_n(A, \_)$ of the left tensor product in degree n with coefficients in the right Λ-module A is defined by

$$\mathrm{T}_n(A, \_)(B) := \mathrm{T}_n(A, B) := \varprojlim_{k, k+n \geqslant 0} \Omega^{k+n} A \overrightarrow{\otimes} \Sigma^k B$$

This produces a homological counterpart to Buchweitz's construction.

Comparing V and T

Theorem

Let A be a right Λ-module. For each $l \in \mathbb{Z}$, there is an epimorphic natural transformation

$$\kappa_l : \mathrm{V}_l(A, \_) \twoheadrightarrow \mathrm{T}_l(A, \_)$$

Comparing V, T, and M

Theorem

For any module A, there is a commutative diagram of connected sequences of functors

$$\begin{tikzcd} & \mathrm{V}_\bullet(A, \_) \arrow[dl, "\kappa"'] \arrow[dr, "\theta"] & \\ \mathrm{T}_\bullet(A, \_) \arrow[rr, "\simeq"] \arrow[dr, "\lambda"'] & & \mathrm{M}_\bullet(\mathrm{Tor}(A, \_)) \arrow[dl, "\tau"] \\ & \mathrm{Tor}(A, \_) & \end{tikzcd}$$

where the horizontal arrow is an isomorphism.

What is Ker κ? A conjecture

The comparison map $\kappa : \mathrm{V}_\bullet(A, \_) \longrightarrow \mathrm{T}_\bullet(A, \_)$ appears to be an algebraic analog of the comparison map from Steenrod-Sitnikov homology to Čech homology. That map is also epic, and its kernel is given by the first derived limit. Based on this analogy, we had

Conjecture (2014)

Ker κ is given by the first derived limit.

A positive answer

The conjecture is true by the following recent result

Theorem (I. Emmanouil and P. Manousaki, 2017)

There is an exact sequence

$$0 \longrightarrow \varprojlim_i{}^1 \operatorname{Tor}_{\bullet+i+1}(A, \Sigma^i \_) \longrightarrow V_\bullet(A, \_) \longrightarrow \mathrm{M}_\bullet(\operatorname{Tor}(A, \_)) \longrightarrow 0$$

Part 5. Definition of torsion

What are we trying to define?

- For any module over any ring, define the torsion submodule, extending classical torsion over commutative domains.

- For any module over any ring, define the cotorsion quotient module.

Classical torsion

Classical torsion is defined over commutative domains:

$$T(A) := \{a \in A \mid \exists r \in R - \{0\},\ ra = 0\}$$

It can be extended to arbitrary rings in more than one way, but we want to simultaneously generalize the notion of 1-torsion.

WHAT IS 1-TORSION?

The 1-torsion $\mathfrak{t}(A)$ of a module A is the kernel of the canonical evaluation map

$$e_A : A \longrightarrow A^{**} : a \mapsto (F_a : f \mapsto f(a))$$

Thus $\mathfrak{t}(A)$ is determined by the exact sequence

$$0 \longrightarrow \mathfrak{t}(A) \longrightarrow A \longrightarrow A^{**}$$

It is defined for any module over any ring and consists of those elements of A on which every linear form on A vanishes. Moreover:

LEMMA

If R is a commutative domain and A is finitely generated, then

$$T(A) = \mathfrak{t}(A)$$

Warning: 1-torsion can be big

For infinite modules, 1-torsion need not coincide with classical torsion.

Example

Let $R := \mathbb{Z}$ and $A := \mathbb{Q}$. Then

$$T(\mathbb{Q}) = \{0\} \quad \text{but} \quad \mathfrak{t}(\mathbb{Q}) = \mathbb{Q}$$

1-Torsion

However, 1-torsion is a ubiquitous concept:

- Stable module theory (Auslander - Bridger, 1969);
- PDE and constructive aspects of linear control systems (Oberst, et al. 1990, . . .);
- Linkage of algebraic varieties (M - Strooker, 2004);
- Algebraic aspects of a question of Reiffen - Vetter (M, 2010).
- Local algebraic geometry, singularity theory, local cohomology, . . .

PRECISE STATEMENT OF THE PROBLEM

Problem Find a common generalization of:

- classical torsion for arbitrary modules over commutative domains, and
- 1-torsion for finitely presented modules over arbitrary rings.

The new definitions should work for arbitrary modules over arbitrary rings.

Classical torsion via localization

Let K be the field of fractions of the commutative domain R and

$$0 \longrightarrow R \longrightarrow K$$

the canonical embedding.

Tensoring a module A with this map, we have the localization map

$$\ell_A : A \cong A \otimes R \to A \otimes_R K$$

Lemma

$T(A) = \operatorname{Ker} \ell_A$.

Observation: In this construction, K is the injective envelope of the ring.

The definition

Let Λ be a ring, A a **right** Λ-module, and

$$0 \longrightarrow \Lambda \longrightarrow I$$

the injective envelope of Λ viewed as a **left** module over itself.

Definition

The torsion of A is defined by the exact sequence

$$0 \longrightarrow \mathfrak{s}(A) \longrightarrow A \otimes \Lambda \longrightarrow A \otimes I$$

We have what we have asked for

Proposition

- *If Λ is a commutative domain, then $\mathfrak{s}$ coincides with classical torsion.*
- *On finitely presented modules over any Λ, $\mathfrak{s}$ coincides with 1-torsion.*

First properties of $\mathfrak{s}$

Theorem

- $\mathfrak{s}$ *is a subfunctor of 1-torsion:* $\mathfrak{s} \subseteq \mathfrak{t}$.
- $\mathfrak{s}$ *preserves filtered colimits (and hence coproducts).*
- $\mathfrak{s}$ *is the largest subfunctor of* $\mathfrak{t}$ *that preserves filtered colimits.*
- $\mathfrak{s}$ *is a radical, i.e.,* $\mathfrak{s}(A/\mathfrak{s}(A)) = \{0\}$ *for any module* A.
- $\mathfrak{s}(A) = 0$ *for any flat module* A.

FURTHER PROPERTIES OF $\mathfrak{s}$

PROPOSITION

The following conditions are equivalent:

A) $\mathfrak{s}$ *is the zero functor (on* `right` *modules);*

B) $\mathfrak{s}$ *preserves epimorphisms;*

C) ${}_\Lambda\Lambda$ *is absolutely pure;*

D) Λ *is* `left` *FP-injective, i.e.,* $\mathrm{Ext}^1_\Lambda(M, \Lambda) = \{0\}$ *for all finitely presented left* Λ*-modules* M*.*

In particular, if Λ *is selfinjective on the left, then* $\mathfrak{s}$ *is the zero functor.*

$\mathfrak{s}$ AND THE REJECT OF FLATS

Let $Rej(A, \mathcal{F})$ be the reject of the class $\mathcal{F}$ of *flats* in the right module A, and $rej(A, \mathcal{F})$ its restriction to finitely presented modules. Then

$$\mathfrak{s} \subseteq Rej(\_, \mathcal{F}) \subseteq \mathfrak{t}$$

Restricting to finitely presented modules, we get equalities. Whence

PROPOSITION

$\mathfrak{s} \simeq \overrightarrow{rej}(\_, \mathcal{F})$, *i.e., the torsion functor is isomorphic to the colimit extension of the reject of flats restricted to finitely presented modules.*

EXERCISE

EXERCISE

Most of the basic results about classical torsion carry over, in one form or another, to the new setting. State such results and prove them.

Part 6. Definition of cotorsion

HOW DO WE DEFINE COTORSION?

In the absence of a classical prototype, we try and dualize the definition of torsion.

Start with a simple question:

- Why is $\mathfrak{s}(A)$ a subset of A?

WHY IS $\mathfrak{s}(A)$ A SUBSET OF A

The answer is obvious: because, by definition, $\mathfrak{s}(A)$ is a subset of $A \otimes \Lambda$ and there is a canonical isomorphism

$$A \otimes \Lambda \xrightarrow{\cong} A$$

Question Is there a "dual" canonical isomorphism?

Answer Yes, there is:

$$\mathrm{Hom}(\Lambda, C) \xrightarrow{\cong} C$$

DUALIZING THE DEFINITION OF TORSION

The torsion of A was defined as the value of the **injective** stabilization of the tensor product functor $A \otimes \_$ on Λ.

Dually, we define the cotorsion of C as the value of the **projective** stabilization of the contravariant Hom functor $\mathrm{Hom}(\_, C)$ on Λ.

DEFINITION OF COTORSION

DEFINITION

Let C be a (left) Λ-module. The cotorsion quotient module of C is

$$\mathfrak{q}(C) := \underline{\mathrm{Hom}(\_, C)}(\Lambda) = \overline{\mathrm{Hom}}(\Lambda, C)$$

Thus $\mathfrak{q} = \overline{\mathrm{Hom}}(\Lambda, \_)$ is a quotient of the identity functor.

FIRST OBSERVATIONS

- The short exact sequences

$$0 \longrightarrow I(\Lambda, C) \longrightarrow (\Lambda, C) \longrightarrow (\overline{\Lambda, C}) \longrightarrow 0$$

give rise to a short exact sequence of endofunctors on Λ-Mod

$$0 \longrightarrow \mathfrak{q}^{-1} \longrightarrow \mathbf{1} \longrightarrow \mathfrak{q} \longrightarrow 0$$

- $\mathfrak{q}$ preserves epimorphisms.
- $\mathfrak{q}$ is finitely presented:

$$(I, \_) \xrightarrow{(\iota, \_)} (\Lambda, \_) \longrightarrow \mathfrak{q} \longrightarrow 0$$

Trace of injectives comes into play

Lemma

Under the canonical isomorphism

$$(\Lambda, C) \cong C : f \mapsto f(1),$$

$I(\Lambda, C)$ *identifies with* $Tr(\mathcal{I}, C)$*, the trace in* C *of the class* $\mathcal{I}$ *of injective* Λ*-modules.*

Proposition

$\mathfrak{q}$ *is a coradical, i.e.,* $\mathfrak{q}(\mathfrak{q}^{-1}(C)) = \{0\}$ *for any* C.

COTORSION MODULES

DEFINITION

The module C is cotorsion if $C \to \mathfrak{q}(C)$ is an isomorphism. In other words, C is cotorsion if no map $\Lambda \to C$ factors through an injective. Equivalently, $Tr(\mathcal{I}, C) = \{0\}$.

EXAMPLE

Any PID which is not a field, viewed as a module over itself, is cotorsion (as it has no nonzero divisible elements).

COTORSION-FREE MODULES

DEFINITION

The module C is cotorsion-free if $C \to \mathfrak{q}(C)$ is the zero map, i.e., any map $\Lambda \to C$ factors through an injective. Equivalently, $Tr(\mathcal{I}, C) = C$.

EXAMPLE

Any injective module is cotorsion-free.

Obviously, $\{0\}$ is the only module which is cotorsion and cotorsion-free.

Expected properties hold

Exercise

Formulate and prove basic properties of cotorsion (Hint: dualize the properties of torsion).

Part 7. The Auslander-Gruson-Jensen functor and friends

THE AUSLANDER-GRUSON-JENSEN FUNCTOR

The Auslander-Gruson-Jensen duality, discovered by Auslander and independently by Gruson and Jensen, is a pair of exact contravariant functors

$$\begin{tikzcd} \mathrm{fp}(\mathrm{mod}(\Lambda^{op}), \mathtt{Ab}) \arrow[rr, bend left, "D"] & & \mathrm{fp}(\mathrm{mod}(\Lambda), \mathtt{Ab}) \arrow[ll, bend left, "D"] \end{tikzcd}$$

each of which interchanges the tensor product and the Hom functor when the fixed argument is a finitely presented module.

AN EXTENSION OF THE AGJ FUNCTOR

There is an exact contravariant functor

$$D_A : \mathrm{fp}(\mathrm{Mod}\,(\Lambda^{op}), \mathtt{Ab}) \to (\mathrm{mod}(\Lambda), \mathtt{Ab})$$

defined by

$$D_A := R_0(\epsilon \circ w),$$

where ϵ is the tensor embedding

$$\epsilon : \mathrm{Mod}\,(\Lambda^{op}) \to (\mathrm{mod}(\Lambda), \mathtt{Ab}) : M \mapsto \_ \otimes M$$

and w is the defect functor. For any representable functor $(M, \_)$

$$D_A(M, \_) = \_ \otimes M$$

As is shown by Dean-Russell (2016), the functor D_A is completely determined by this property and by being exact.

THE AGJ FUNCTOR AND FRIENDS

THEOREM (S. DEAN - J. RUSSELL, 2016)

The functor

$$D_A : \mathrm{fp}(\mathrm{Mod}\,(\Lambda^{op}), \mathcal{A}b) \to (\mathrm{mod}(\Lambda), \mathcal{A}b)$$

admits a left adjoint D_L and a right adjoint D_R, both of which are fully faithful. The functors D_R and D_A restrict to the AGJ duality D on the full subcategories of pp-functors.

GENERAL PICTURE

The foregoing statement is part of the following diagram of functors

$$
\begin{tikzcd}
\mathrm{fp}(\mathrm{Mod}\,(\Lambda^{op}), \mathtt{Ab}) \arrow[rr, "D_A"] \arrow[dr, bend right=10, "w"'] & & (\mathrm{mod}(\Lambda), \mathtt{Ab}) \arrow[ll, bend right, "D_L"'] \arrow[ll, bend left, "D_R"] \arrow[dl, bend right=10, "\mathrm{ev}_\Lambda"] \\
& \mathrm{Mod}\,(\Lambda^{op}) \arrow[ul, bend left, "\mathsf{Y}"] \arrow[ul, bend right, "L^0\mathsf{Y}"'] \arrow[ur, bend left, "R_0\epsilon"] \arrow[ur, bend right, "\epsilon"'] &
\end{tikzcd}
$$

SENDING COTORSION TO TORSION

THEOREM

For any module B

$$D_A\overline{\mathrm{Hom}}(B, \_) \simeq \_ \overrightarrow{\otimes} B$$

COROLLARY

Spcializing to $B := \Lambda$, we have

$$D_A(\mathfrak{q}) \simeq \mathfrak{s}$$

(dexterity changes). Equivalently,

$$D_A(Tr(\mathcal{I}, \_)^{-1}) \simeq \overrightarrow{rej}(\_, \mathcal{F})$$

Going back: sending torsion to cotorsion

Proposition

For any pure injective left Λ-module M,

$$\overline{\mathrm{Hom}}(M, \_) \simeq D_L(\_ \overrightarrow{\otimes} M)$$

Corollary

If ${}_\Lambda\Lambda$ is pure injective, then $\mathfrak{q} \simeq D_L(\mathfrak{s})$.

GOING BACK: ANOTHER OPTION

THEOREM

Suppose the injective envelope of Λ is finitely presented. Then the notions of torsion and cotorsion are dual. More precisely, the right adjoint

$$D_R : (\mathrm{mod}(\Lambda), \mathcal{A}b) \to \mathrm{fp}(\mathrm{Mod}\,(\Lambda^{op}), \mathcal{A}b)$$

of D_A carries the torsion functor to the cotorsion functor, i.e.,

$$D_R(\mathfrak{s}) \simeq \mathfrak{q}$$

COROLLARY

Let Λ be an artin algebra. Then $D_R(\mathfrak{s}) \simeq \mathfrak{q}$.

THE AUSLANDER-REITEN FORMULA FOR ARBITRARY MODULES, AGAIN

The foregoing isomorphisms are between functors, with no apparent connections between their arguments. We can do better (slide 36):

$$D_{\mathbf{J}}(\mathfrak{s}(A)) \simeq \mathfrak{q}(D_{\mathbf{J}}(A))$$

Thus, the dual of the torsion of a module is the cotorsion of the dual of the module.

EXCHANGE FORMULA

EXAMPLE

Let Λ be any ring, $R := \mathbb{Z}$, and $\mathbf{J} := \mathbb{Q}/\mathbb{Z}$. Set $(\_)^+ := D_{\mathbf{J}}(\_)$. Then

$$\mathfrak{s}(A)^+ \simeq \mathfrak{q}(A^+)$$