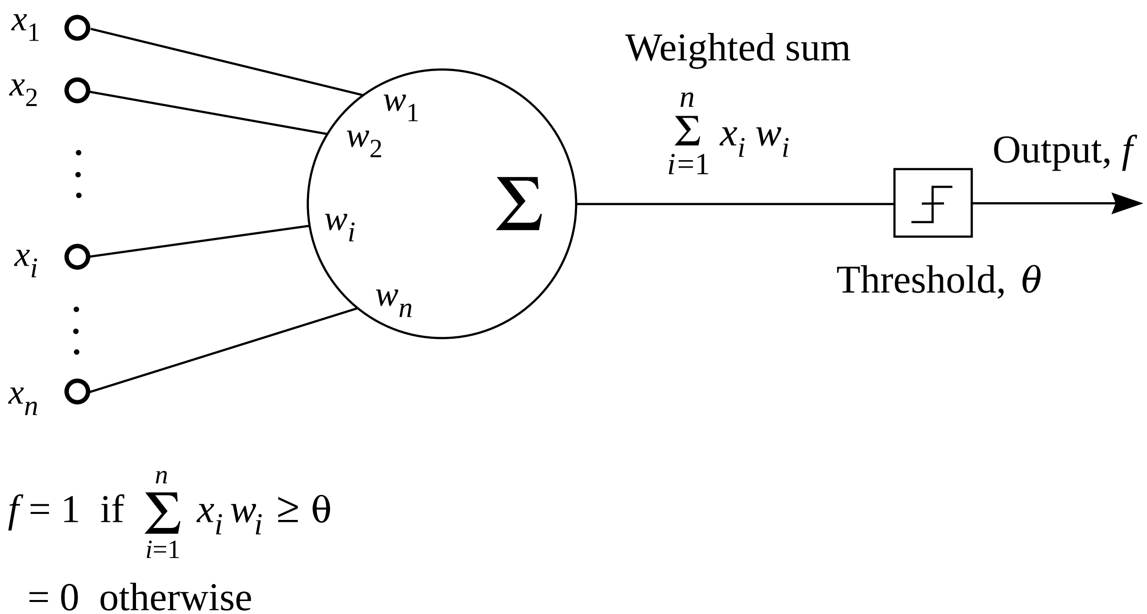
NEURAL NETWORKS II

Introduction

- This lecture builds on the description in the previous lecture to establish how to train neural networks.
- We will work out a general approach.
- We will then give three particular versions that are commonly used.
- We start with a quick recap.

Recap



– Output is 1 if

$$s = X \cdot W > \theta$$

– Output is 0 otherwise

Gradient descent methods

- A common way to train a TLU is through an error function.
- We define:

$$\epsilon = \sum_{X \in \Theta} (d_i(X_i) - f_i(X_i))^2$$

- where:
  - $d_i(X_i)$ is the outcome we want for X_i;
  - $f_i(X_i)$ is the outcome we get.
- Often we write these functions as d_i and f_i.
- We then minimise ϵ

- If θ is rolled into the weights, then the value of ϵ depends on the weights.
- (Since these determine the value of f_i.)
- We minimise by looking at the gradient of ϵ with respect to the weights…
- …and then altering the weights to move ϵ down the gradient.
- Eventually this *gradient descent* will take us down to the minimum value of ϵ.

- The computation of ϵ is complicated by the fact that its value depends on *all* the X_i in Θ.
- Often it is easier to do the calculation for one X_i, adjust the weights to move down the gradient, and then start over with another X_j.
- Thus we do the learning incrementally, taking each member of Θ in an order Σ.
- The incremental version only ever approximates the result of doing it "properly" (the batch way), but often it is a good approximation.
- Here we will just look at the incremental version.

- When we have a single input vector X, with output f and desired output d, the error is:
$$\epsilon = (d - f)^2$$
- The gradient of ϵ with respect to the weights is
$$\frac{\partial \epsilon}{\partial W}$$
and
$$\frac{\partial \epsilon}{\partial W} = \left[\frac{\partial \epsilon}{\partial w_1}, \frac{\partial \epsilon}{\partial w_2}, \ldots, \frac{\partial \epsilon}{\partial w_{n+1}}\right]$$

- Since ϵ depends on W through

$$s = X \cdot W$$

it follows that:

$$\frac{\partial \epsilon}{\partial W} = \frac{\partial \epsilon}{\partial s}\frac{\partial s}{\partial W}$$

- Since:

$$\frac{\partial s}{\partial W} = X$$

it follows that:

$$\frac{\partial \epsilon}{\partial W} = \frac{\partial \epsilon}{\partial s} X$$

- Furthermore we can write:

$$\frac{\partial \epsilon}{\partial s} = -2(d - f)\frac{\partial f}{\partial s}$$

  and so:

$$\frac{\partial \epsilon}{\partial W} = -2(d - f)\frac{\partial f}{\partial s}X$$

- This seems to give us a way of working out what the gradient is.
- However, we have a problem.

- The problem is that the TLU output f, cannot be differentiated.
- Most times we vary s a little we get no change in f.
- Sometimes, though, we get a big change (from 0 to 1 or vice-versa).
- There are several ways around this.
  - Ignore the threshold and set $f = s$.
  - Replace the threshold function with something we can differentiate or otherwise find the gradient of.
- We will look at both of these.

The Widrow-Hoff procedure

- Let's try and adjust the weights so that:
  - Every training vector labelled with a 1 produces a dot product of 1; and
  - Every training vector labelled with a 0 produces a dot product of -1.
- Then, with

$$f = s$$

the incremental squared error is:

$$\epsilon = (d - f)^2 = (d - s)^2$$

and

$$\frac{\partial f}{\partial s} = 1$$

- This makes the gradient:

$$\frac{\partial \epsilon}{\partial W} = -2(d - f)X$$

- If we want to then move the weight vector down the gradient, we can set the new value of the weight vector as:

$$W := W + c(d - f)X$$

- The factor of 2 is combined into the *learning rate parameter* c.
- As always this controls the speed of the adjustment by determining the fraction of X added to W.

- Whenever the error:

$$(d - f)$$

is positive, then we add a fraction of the input into the weight.

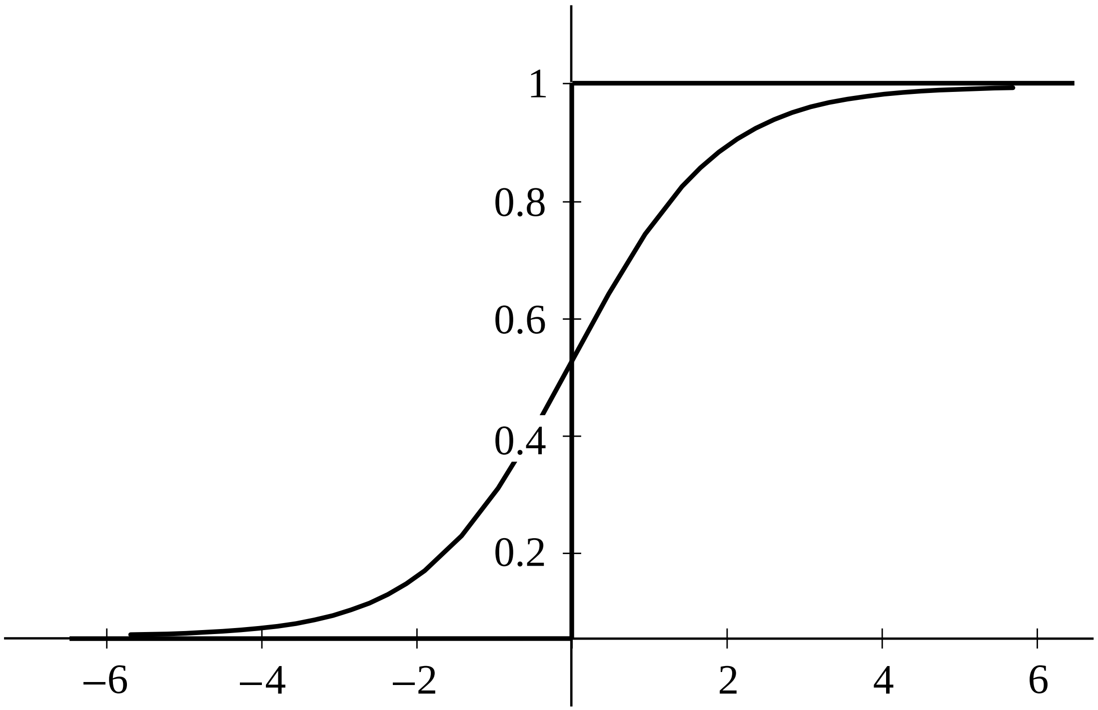
- This increases $X \cdot W$, and so decreases

$$(d - f)$$

- If the error is negative we subtract a fraction of the input and reverse the effect.
- Once we have found the best set of weights, we can go back to using the threshold function.

The generalised Delta procedure

- Another way to handle the threshold function is to replace it with something we can differentiate.



- This function is known as a *sigmoid*:

$$f(s) = \frac{1}{1 + e^{-s}}$$

- With this function, we have the partial derivative:

$$\frac{\partial f}{\partial s} = f(1 - f)$$

- Since

$$\frac{\partial \epsilon}{\partial W} = -2(d - f)\frac{\partial f}{\partial s}X$$

  we have:

$$\frac{\partial \epsilon}{\partial W} = -2(d - f)f(1 - f)X$$

- This gives us another rule for changing weights:

$$W := W + c(d - f)f(1 - f)X$$

- This compares to the Widrow-Hoff procedure as follows:
  - In W-H, d is 1 or -1. In generalised Delta it is 1 or 0.
  - In W-H, f is equal to s. In generalised Delta, f is the output of the sigmoid function.
  - Generalised Delta has the extra term $f(1 - f)$
- With the sigmoid, $f(1 - f)$ varies in value from 0 to 1.
- It has value 0 when f is 0 or 1.
- It has maximum value of 0.25 when f has value 0.5 (and the input to the sigmoid is 0).

- One can think of the sigmoid as a “fuzzy boundary”.
- When the input is a long way from the boundary, $f(1 - f)$ has a value close to 0.
- Thus hardly any adjustment is made to the weights.
- When the input is closer to the boundary, then weight changes are more significant.
- These changes are always to reduce the error.
- Once the weights are established, we can go back to using the step function.

A general approach

- Both these techniques have done the same thing.
- They have replaced something we couldn't find the slope of with something we could.
- We could do the same with a gradient function (as we will in the homework).
- This obviously trains the weights approximately.
- However, it seems that the approximation is often good enough.
- In any case, we are interested in performance on non-training examples.

The error-correction procedure

- Another approach keeps the original threshold function.
- We then forget about differentiation and just adjust the weights when the TLU gives a classification error.
- In other words we make a change when:

  $$(d - f)$$

  has value 1 or -1.
- This time the weight change rule is:

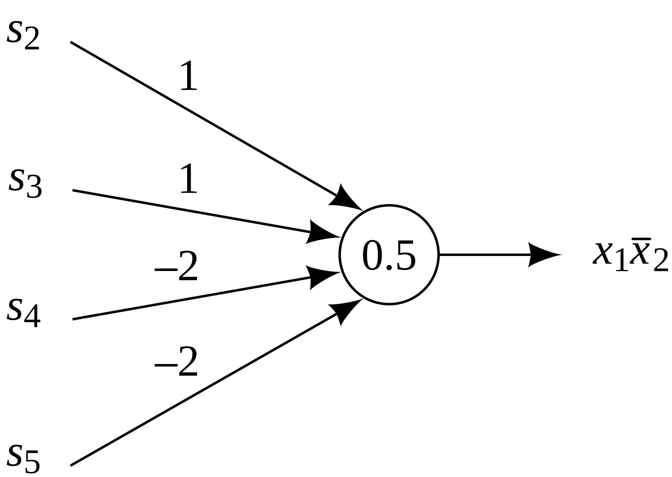
  $$w := W + c(d - f)X$$

- Just as before, the change tends to reduce the error.

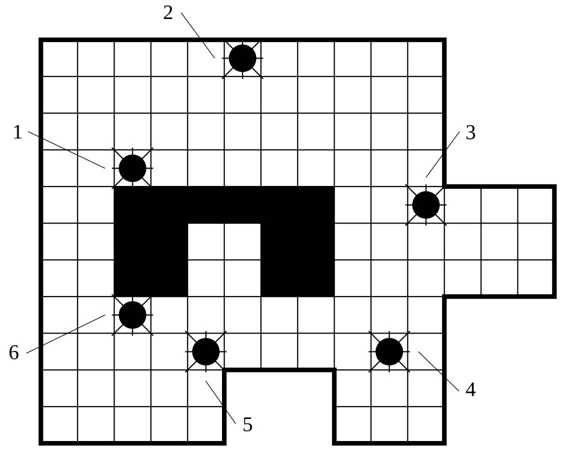
- Comparing this with Widrow-Hoff, we note that both d and f are either 0 or 1.
- Whereas in W-H, d is 1 or -1 and $f = s$.
- It is possible to prove that if there is a W that gives a correct output for all $X \in \Theta$,
- Then after a finite number of adjustments, this error-correction procedure will find this weight vector.
- Thus the process will terminate, making no more weight adjustments.
- For nonlinearly separable sets of input vectors, the procedure will not terminate (as opposed to W-H and generalised Delta).

- Since (as we saw last lecture) the rules/network for the boundary following robot are linearly separable functions…
- …we can use any of these procedures to learn the weights for a TLU to implement these functions, such as:

$$(s_2 + s_3)\overline{s_4 s_5}$$



- A suitable training set for training this TLU is



Input number	Sensory vector	$x_1\overline{x_2}$ (move east)
1	00001100	0
2	11100000	1
3	00100000	1
4	00000000	0
5	00001000	0
6	01100000	1

- Let's consider training using the error-correction procedure:

Summary

- In this lecture we looked at methods for training TLUs.
- All the methods were *gradient descent*—they adjusted weights to reduce the error, step-by-step.
- They differed in what they used for the threshold function.
- Widrow-Hoff ignores it and sets $f = s$.
- Generalised-delta uses a function that can be differentiated.
- Error-correction uses the step function.