PREDICATE LOGIC

First-Order Logic

- Aim of this lecture:
  to introduce *first-order predicate logic*.
- *More expressive* than propositional logic.
- Consider the following argument:
  - *all monitors are ready;*
  - *X12 is a monitor;*
  - therefore *X12 is ready*.
- Sense of this argument *cannot* be captured in propositional logic.
- Propositional logic is too *coarse grained* to allow us to represent and reason about this kind of statement.

Syntax

- We shall now introduce a generalisation of propositional logic called first-order logic (FOL). This new logic affords us much greater expressive power.
- **Definition:** The alphabet of FOPL contains:

  1. a set of *constants*;
  2. a set of *variables*;
  3. a set of *function symbols*;
  4. a set of *predicates symbols*;
  5. the connectives $\vee$, $\neg$;
  6. the *quantifiers* $\forall$, $\exists$, $\exists_1$;
  7. the punctuation symbols), (.

Terms

- The basic components of FOL are called *terms*.
- Essentially, a term is an object that *denotes* some object other than $\top$ or $\perp$.
- The simplest kind of term is a *constant*.
- A value such as 8 is a constant.
- The *denotation* of this term is the number 8.
- Note that a constant and the number it denotes are different!
- Aliens don't write "8" for the number 8, and nor did the Romans.

- The second simplest kind of term is a *variable*.
- A variable can stand for anything in the *domain of discourse*.
- The domain of discourse (usually abbreviated to domain) is the set of all objects under consideration.
- Sometimes, we assume the set contains "everything".
- Sometimes, we explicitly *give* the set, and *state* what variables/constants can stand for.

Functions

- We can now introduce a more complex class of terms — *functions*.
- The idea of functional terms in logic is similar to the idea of a function in programming: recall that in programming, a function is a procedure that takes some arguments, and *returns a value*.

  In Modula-2:

  ```
  PROCEDURE f(a1:T1; ...; an:Tn) : T;
  ```

  this function takes n arguments; the first is of type `T1`, the second is of type `T2`, and so on. The function returns a value of type `T`.
- In FOL, we have a set of *function symbols*; each symbol corresponds to a particular function. (It denotes some function.)

- Each function symbol is associated with a number called its *arity*. This is just the number of arguments it takes.
- A *functional term* is built up by *applying* a function symbol to the appropriate number of terms.
- Formally …

  Definition: Let f be an arbitrary function symbol of arity n. Also, let $\tau_1, \ldots, \tau_n$ be terms. Then

  $$f(\tau_1, \ldots, \tau_n)$$

  is a functional term.

- All this sounds complicated, but isn't. Consider a function *plus*, which takes just two arguments, each of which is a number, and returns the first number added to the second.

  Then:

  - *plus*$(2, 3)$ is an acceptable functional term;
  - *plus*$(0, 1)$ is acceptable;
  - *plus*$($*plus*$(1, 2), 4)$ is acceptable;
  - *plus*$($*plus*$($*plus*$(0, 1), 2), 4)$ is acceptable;

- In maths, we have many functions; the obvious ones are

$$+ \quad - \quad / \quad * \quad \sqrt{\quad\quad} \quad \sin \quad \cos \quad \ldots$$

- The fact that we write

$$2 + 3$$

instead of something like

$$\textit{plus}(2, 3)$$

is just convention, and is not relevant from the point of view of logic; all these are functions in exactly the way we have defined.

- Using functions, constants, and variables, we can build up *expressions*, e.g.:

$$(x + 3) * \sin 90$$

(which might just as well be written

$$times(plus(x, 3), sin(90))$$

for all it matters.)

Predicates

- In addition to having terms, FOL has *relational operators*, which capture *relationships* between objects.
- The language of FOL contains *predicate symbols*.
- These symbols stand for *relationships between objects*.
- Each predicate symbol has an associated *arity* (number of arguments).
- **Definition:** Let P be a predicate symbol of arity n, and $\tau_1, \ldots, \tau_n$ are terms.

  Then

  $$P(\tau_1, \ldots, \tau_n)$$

  is a predicate, which will either be $\top$ or $\bot$ under some interpretation.

- EXAMPLE. Let *gt* be a predicate symbol with the intended interpretation 'greater than'. It takes two arguments, each of which is a natural number.

  Then:

  – $gt(4,3)$ is a predicate, which evaluates to $\top$;

  – $gt(3,4)$ is a predicate, which evaluates to $\perp$.

- The following are standard mathematical predicate symbols:

$$> < = \geq \leq \neq \ \ldots$$

- The fact that we are normally write $x > y$ instead of $gt(x,y)$ is just convention.

- We can build up more complex predicates using the connectives of propositional logic:

$$(2 > 3) \wedge (6 = 7) \vee (\sqrt{4} = 2)$$

- So a predicate just expresses a relationship between some values.
- What happens if a predicate contains *variables*: can we tell if it is true or false?

  Not usually; we need to know an *interpretation* for the variables.
- A predicate that contains no variables is a proposition.

- Predicates of arity 1 are called *properties*.
- EXAMPLE. The following are properties:

$$\begin{array}{l} Man(x) \\ Mortal(x) \\ Malfunctioning(x). \end{array}$$

- We interpret $P(x)$ as saying x is in the set P.
- Predicate that have arity 0 (i.e., take no arguments) are called *primitive propositions*.

  These are identical to the primitive propositions we saw in propositional logic.

Quantifiers

- We now come to the central part of first order logic: *quantification*.
- Consider trying to represent the following statements:
  - *all men have a mother*;
  - *every positive integer has a prime factor*.
- We can't represent these using the apparatus we've got so far; we need *quantifiers*.

- We use three quantifers:

  $\forall$ — *the universal quantifier*;
  is read 'for all...'

  $\exists$ — *the existential quantifier*;
  is read 'there exists...'

  $\exists_1$ — *the unique quantifier*;
  is read 'there exists a unique...'

- The simplest form of quantified formula is as follows:

$$\textit{quantifier variable} \cdot \textit{predicate}$$

where

  - *quantifier* is one of $\forall$, $\exists$, $\exists_1$;
  - *variable* is a variable;
  - and *predicate* is a predicate.

Examples

- $\forall x \cdot Man(x) \Rightarrow Mortal(x)$

  'For all x, if x is a man, then x is mortal.'

  (i.e. all men are mortal)

- $\forall x \cdot Man(x) \Rightarrow \exists_1 y \cdot Woman(y) \wedge MotherOf(x, y)$

  'For all x, if x is a man, then there exists exactly one y such that y is a woman and the mother of x is y.'

  (i.e., every man has exactly one mother).

- $\exists m \cdot \mathit{Monitor}(m) \wedge \mathit{MonitorState}(m, \mathit{ready})$

  'There exists a monitor that is in a ready state.'

- $\forall r \cdot \mathit{Reactor}(r) \Rightarrow \exists_1 t \cdot (100 \leq t \leq 1000) \wedge \mathit{temp}(r) = t$

  'Every reactor will have a temperature in the range 100 to 1000.'

- $\exists n \cdot posInt(n) \wedge n = (n * n)$
  'Some positive integer is equal to its own square.'
- $\exists c \cdot ECCountry(c) \wedge Borders(c, Albania)$
  'Some EC country borders Albania.'
- $\forall m, n \cdot Person(m) \wedge Person(n) \Rightarrow \neg Superior(m, n)$
  'No person is superior to another.'
- $\forall m \cdot Person(m) \Rightarrow \neg \exists n \cdot Person(n) \wedge Superior(m, n)$
  Ditto.

Domains & Interpretations

- Suppose we have a formula $\forall x \cdot P(x)$.

  What does x *range over*?

  Physical objects, numbers, people, times, $\ldots$?
- Depends on the *domain* that we intend.
- Often, we *name* a domain to make our intended interpretation clear.

- Suppose our intended interpretation is the +ve integers. Suppose $>, +, *, \ldots$ have the usual mathematical interpretation.
- Is this formula *satisfiable* under this interpretation?

$$\exists n \cdot n = (n * n)$$

- Now suppose that our domain is all living people, and that $*$ means "is the child of".
- Is the formula satisfiable under this interpretation?

Comments

- Note that universal quantification is similar to conjunction. Suppose the domain is the numbers $\{2, 4, 6\}$. Then

$$\forall n \cdot \mathit{Even}(n)$$

is the same as

$$\mathit{Even}(2) \wedge \mathit{Even}(4) \wedge \mathit{Even}(6).$$

- Existential quantification is the same as *disjunction*. Thus with the same domain,

$$\exists n \cdot \mathit{Even}(n)$$

is the same as

$$\mathit{Even}(2) \vee \mathit{Even}(4) \vee \mathit{Even}(6).$$

- The universal and existential quantifiers are in fact *duals* of each other:

$$\forall x \cdot P(x) \;\Leftrightarrow\; \neg\exists x \cdot \neg P(x)$$

Saying that everything has some property is the same as saying that there is nothing that does not have the property.

$$\exists x \cdot P(x) \;\Leftrightarrow\; \neg\forall x \cdot \neg P(x)$$

Saying that there is something that has the property is the same as saying that its not the case that everything doesn't have the property.

Decidability

- In propositional logic, we saw that some formulae were tautologies — they had the property of being true under all interpretations.
- We also saw that there was a procedure which could be used to tell whether any formula was a tautology — this procedure was the truth-table method.
- A formula of FOL that is true under all interpretations is said to be *valid*.
- So in theory we could check for validity by writing down all the possible interpretations and looking to see whether the formula is true or not.

- Unfortuately in general we can't use this method.
- Consider the formula:

$$\forall n \cdot Even(n) \Rightarrow \neg Odd(n)$$

- There are an infinite number of interpretations.
- Is there any other procedure that we can use, that will be guaranteed to tell us, in a finite amount of time, whether a FOL formula is, or is not, valid?
- The answer is *no*.
- FOL is for this reason said to be *undecidable*.

Proof in FOL

- Proof in FOL is similar to PL; we just need an extra set of rules, to deal with the quantifiers.
- FOL *inherits* all the rules of PL.
- To understand FOL proof rules, need to understand *substitution*.
- The most obvious rule, for $\forall$-E.

  Tells us that if everything in the domain has some property, then we can infer that any *particular* individual has the property.

$$\frac{\vdash \forall x \cdot \phi(x);}{\vdash \phi(a)} \; \forall\text{-E} \quad \text{for any } a \text{ in the domain}$$

  Going from *general* to *specific*.

- Example 1.

  Let's use $\forall$-E to get the Socrates example out of the way.

$$
\begin{array}{l}
Man(s); \forall x \cdot Man(x) \Rightarrow Mortal(x) \\
\qquad \vdash Mortal(s)
\end{array}
$$

$$
\begin{array}{lll}
1. & Man(s) & \text{Given} \\
2. & \forall x \cdot Man(x) \Rightarrow Mortal(x) & \text{Given} \\
3. & Man(s) \Rightarrow Mortal(s) & 2, \forall\text{-E} \\
4. & Mortal(s) & 1, 3, \Rightarrow\text{-E}
\end{array}
$$

- We can also go from the general to the slightly less specific!

$$\frac{\vdash \forall x \cdot \phi(x);}{\vdash \exists x \cdot \phi(x)} \; \exists\text{-I(1)} \text{ if domain not empty}$$

Note the *side condition*.

The $\exists$ quantifier *asserts the existence* of at least one object.

The $\forall$ quantifier does not.

- We can also go from the very specific to less specific.

$$\frac{\vdash \phi(a);}{\vdash \exists x \cdot \phi(x)} \; \exists\text{-I(2)}$$

- In other words once we have a concrete example, we can infer there exists something with the property of that example.

- We often informally make use of arguments along the lines…
  1. We know somebody is the murderer.
  2. Call this person a.
  3. …

  (Here, a is called a *Skolem constant*.)
- We have a rule which allows this, but we have to be careful how we use it!

$$\frac{\vdash \exists x \cdot \phi(x);}{\vdash \phi(a)} \; \exists\text{-E} \quad a \text{ doesn't occur elsewhere}$$

- Here is an *invalid* use of this rule:

1.	$\exists x \cdot Boring(x)$	Given
2.	$Lecture(AI)$	Given
3.	$Boring(AI)$	1, $\exists$-E

- (The conclusion may be true, the argument isn't sound.)

- Another kind of reasoning:
  - Let a be arbitrary object.
  - $\ldots$ (some reasoning) $\ldots$
  - Therefore a has property ϕ
  - Since a was arbitrary, it must be that every object has property a.
- Common in mathematics:

  Consider a positive integer n $\ldots$ so n is either a prime number or divisible by a smaller prime number $\ldots$ so every positive integer is either a prime number or divisible by a smaller prime number.

- If we are careful, we can also use this kind of reasoning:

$$\frac{\vdash \phi(a);}{\vdash \forall x \cdot \phi(x)} \, \forall\text{-I} \quad a \text{ is arbitrary}$$

- Invalid use of this rule:

$$\begin{array}{lll} 1. & Boring(AI) & \text{Given} \\ 2. & \forall x \cdot Boring(x) & 1, \forall\text{-I} \end{array}$$

- Example 2:

  1. Everybody is either happy or rich.
  2. Simon is not rich.
  3. Therefore, Simon is happy.

  Predicates:

  – $H(x)$ means x is happy;
  – $R(x)$ means x is rich.

- Formalisation:

$$\forall x. H(x) \lor R(x); \neg R(Simon) \vdash H(Simon)$$

1.	$\forall x.H(x) \vee R(x)$	Given
2.	$\neg R(Simon)$	Given
3.	$H(Simon) \vee R(Simon)$	1, $\forall$-E
4.	$\neg H(Simon) \Rightarrow R(Simon)$	3, defn $\Rightarrow$
5.	$\neg H(Simon)$	Ass |
6.	$R(Simon)$	4, 5, $\Rightarrow$-E |
7.	$R(Simon) \wedge \neg R(Simon)$	2, 6, $\wedge$-I |
8.	$\neg\neg H(Simon)$	5, 7, $\neg$-I
9.	$H(Simon) \Leftrightarrow \neg\neg H(Simon)$	PL axiom
10.	$(H(Simon) \Rightarrow \neg\neg H(Simon))$ $\wedge(\neg\neg H(Simon) \Rightarrow H(Simon))$	9, defn $\Leftrightarrow$
11.	$\neg\neg H(Simon) \Rightarrow H(Simon)$	10,$\wedge$-E
12.	$H(Simon)$	8, 11, $\Rightarrow$-E

Summary

- This lecture looked at predicate (or first order) logic.
- Predicate logic is a generalisation of propositional logic.
- The generalisation requires the use of quantifiers, and these need special rules for handling them when doing inference.
- We looked at how the proof rules for propositional logic need to be extended to handle quantifiers.