PROPOSITIONAL LOGIC

What is a Logic?

- When most people say 'logic', they mean either *propositional logic* or *first-order predicate logic*.
- However, the precise definition is quite broad, and literally hundreds of logics have been studied by philosophers, computer scientists and mathematicians.
- Any 'formal system' can be considered a logic if it has:
  - a well-defined *syntax*;
  - a well-defined *semantics*; and
  - a well-defined *proof-theory*.

- The *syntax* of a logic defines the syntactically acceptable objects of the language, which are properly called *well-formed formulae* (wff). (We shall just call them formulae.)
- The *semantics* of a logic associate each formula with a *meaning*.
- The *proof theory* is concerned with manipulating formulae according to certain rules.

Propositional Logic

- The simplest, and most abstract logic we can study is called *propositional logic*.
- **Definition:** A *proposition* is a statement that can be either *true* or *false*; it must be one or the other, and it cannot be both.
- EXAMPLES. The following are propositions:
  - the reactor is on;
  - the wing-flaps are up;
  - Marvin K Mooney is president.

  whereas the following are not:
  - are you going out somewhere?
  - 2+3

- It is possible to determine whether any given statement is a proposition by prefixing it with:

  It is true that . . .

  and seeing whether the result makes grammatical sense.
- We now define *atomic* propositions. Intuitively, these are the set of smallest propositions.
- **Definition:** An *atomic proposition* is one whose truth or falsity does not depend on the truth or falsity of any other proposition.
- So all the above propositions are atomic.

- Now, rather than write out propositions in full, we will abbreviate them by using *propositional variables*.
- It is standard practice to use the lower-case roman letters

$$p, q, r, \ldots$$

  to stand for propositions.
- If we do this, we must define what we mean by writing something like:

  Let p be *Marvin K Mooney is president*.
- Another alternative is to write something like *reactor_is_on*, so that the interpretation of the propositional variable becomes obvious.

The Connectives

- Now, the study of atomic propositions is pretty boring. We therefore now introduce a number of *connectives* which will allow us to build up *complex propositions*.
- The connectives we introduce are:

$\wedge$	and (& or .)
$\vee$	or ($\mid$ or +)
$\neg$	not ($\sim$)
$\Rightarrow$	implies ($\supset$ or $\rightarrow$)
$\Leftrightarrow$	iff ($\leftrightarrow$)

- (Some books use other notations; these are given in parentheses.)

And

- Any two propositions can be combined to form a third proposition called the *conjunction* of the original propositions.
- **Definition:** If p and q are arbitrary propositions, then the *conjunction* of p and q is written

$$p \wedge q$$

  and will be true iff both p and q are true.

- We can summarise the operation of $\wedge$ in a *truth table*. The idea of a truth table for some formula is that it describes the behaviour of a formula under all possible interpretations of the primitive propositions the are included in the formula.
- If there are n different atomic propositions in some formula, then there are 2^n different lines in the truth table for that formula. (This is because each proposition can take one 1 of 2 values — *true* or *false*.)
- Let us write T for truth, and F for falsity. Then the truth table for $p \wedge q$ is:

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Or

- Any two propositions can be combined by the word 'or' to form a third proposition called the *disjunction* of the originals.
- **Definition:** If p and q are arbitrary propositions, then the *disjunction* of p and q is written

$$p \vee q$$

  and will be true iff either p is true, or q is true, or both p and q are true.

- The operation of $\vee$ is summarised in the following truth table:

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

- Note that this 'or' is a little different from the usual meaning we give to 'or' in everyday language.

If... Then...

- Many statements, particularly in mathematics, are of the form:

  if p *is true then* q *is true.*

  Another way of saying the same thing is to write:

  p *implies* q.

- In propositional logic, we have a connective that combines two propositions into a new proposition called the *conditional*, or *implication* of the originals, that attempts to capture the sense of such a statement.

- **Definition:** If p and q are arbitrary propositions, then the *conditional* of p and q is written

$$p \Rightarrow q$$

and will be true iff either p is false or q is true.

- The truth table for $\Rightarrow$ is:

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

- The $\Rightarrow$ operator is the hardest to understand of the operators we have considered so far, and yet it is extremely important.
- If you find it difficult to understand, just remember that the $p \Rightarrow q$ means 'if p is true, then q is true'.

  If p is false, then we don't care about q, and by default, make $p \Rightarrow q$ evaluate to T in this case.
- Terminology: if ϕ is the formula $p \Rightarrow q$, then p is the *antecedent* of ϕ and q is the *consequent*.

Iff

- Another common form of statement in maths is:

  p *is true if, and only if,* q *is true*.

- The sense of such statements is captured using the *biconditional* operator.
- **Definition:** If p and q are arbitrary propositions, then the *biconditional* of p and q is written:

  $$p \Leftrightarrow q$$

  and will be true iff either:

  1. p and q are both true; or
  2. p and q are both false.

- The truth table for $\Leftrightarrow$ is:

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

- If $p \Leftrightarrow q$ is true, then p and q are said to be *logically equivalent*. They will be true under exactly the same circumstances.

Not

- All of the connectives we have considered so far have been *binary*: they have taken *two* arguments.
- The final connective we consider here is *unary*. It only takes *one* argument.
- Any proposition can be prefixed by the word ‘not’ to form a second proposition called the *negation* of the original.

- **Definition:** If p is an arbitrary proposition then the *negation* of p is written

$$\neg p$$

and will be true iff p is false.

- Truth table for $\neg$:

p	$\neg p$
F	T
T	F

Comments

- We can *nest* complex formulae as deeply as we want.
- We can use *parentheses* i.e.,),(, to *disambiguate* formulae.
- EXAMPLES. If p, q, r, s and t are atomic propositions, then all of the following are formulae:
  - $p \wedge q \Rightarrow r$
  - $p \wedge (q \Rightarrow r)$
  - $(p \wedge (q \Rightarrow r)) \vee s$
  - $((p \wedge (q \Rightarrow r)) \vee s) \wedge t$

  whereas none of the following is:
  - $p \wedge$
  - $p \wedge q)$
  - $p \neg$

Syntax

- We have already informally introduced propositional logic; we now define it formally.
- To define the syntax, we must consider what symbols can appear in formulae, and the rules governing how these symbols may be put together to make acceptable formulae.
- **Definition:** Propositional logic contains the following symbols:
  1. A set of *primitive propositions*, $\Phi = \{p, q, r \ldots\}$.
  2. The unary logical connective '$\neg$' (not), and binary logical connective '$\vee$' (or).
  3. The punctuation symbols ')' and '('.
- The remaining logical connectives ($\wedge$, $\Rightarrow$, $\Leftrightarrow$) will be introduced as abbreviations.

- We now look at the rules for putting formulae together.
- **Definition:** The set $\mathcal{W}$, of (well formed) formulae of propositional logic, is defined by the following rules:
  1. If $p \in \Phi$, then $p \in \mathcal{W}$.
  2. If $\phi \in \mathcal{W}$, then:
  $$\neg\phi \in \mathcal{W}$$
  $$(\phi) \in \mathcal{W}$$
  3. If $\phi \in \mathcal{W}$ and $\psi \in \mathcal{W}$, then $\phi \vee \psi \in \mathcal{W}$.

- The remaining connectives are defined by:

$$\begin{aligned} \phi \wedge \psi &= \neg(\neg\phi \vee \neg\psi) \\ \phi \Rightarrow \psi &= (\neg\phi) \vee \psi \\ \phi \Leftrightarrow \psi &= (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \end{aligned}$$

- This concludes the formal definition of syntax.

Semantics

- We now look at the more difficult issue of *semantics*, or *meaning*.
- What does a proposition *mean*?
- That is, when we write

  It is raining.

  what does it mean?

  From the point of view of logic, this statement is a *proposition*: something that is either $\top$ or $\perp$.
- *The meaning of a primitive proposition is thus either* $\top$ *or* $\perp$*.*
- In the same way, the meaning of a formula of propositional logic is either $\top$ or $\perp$.

- QUESTION: How can we tell whether a formula is $\top$ or $\bot$?
- For example, consider the formula

$$(p \wedge q) \Rightarrow r$$

  Is this $\top$?
- The answer must be: *possibly*. It depends on your *interpretation* of the primitive propositions p, q and r.
- The notion of an interpretation is easily formalised.
- **Definition:** An *interpretation* for propositional logic is a function

$$\pi : \Phi \mapsto \{T, F\}$$

  which assigns T (true) or F (false) to every primitive proposition.

Tautologies & Consistency

- When we consider formulae in terms of interpretations, it turns out that some have interesting properties.
- **Definition:**
  1. A formula is a *tautology* iff it is true under *every* valuation;
  2. A formula is *consistent* iff it is true under *at least one* valuation;
  3. A formula is *inconsistent* iff it is not made true under *any* valuation.
- A tautology is said to be *valid*.
- A consistent formula is said to be *satisfiable*.
- An inconsistent formula is said to be *unsatisfiable*.

- **Theorem:** ϕ is a tautology iff $\neg\phi$ is unsatisfiable.
- Now, each line in the truth table of a formula corresponds to a valuation.
- So, we can use truth tables to determine whether or not formulae are tautologies.
- If every line in the truth tabel has value T, the the formula is a tautology.
- Also use truth-tables to determine whether or not formulae are *consistent*.

- To check for consistency, we just need to find *one* valuation that satisfies the formula.
- If this turns out to be the first line in the truth-table, we can stop looking immediately: we have a *certificate* of satisfiability.
- To check for validity, we need to examine *every* line of the truth-table.

  No short cuts.
- The lesson? *Checking satisfiability is easier than checking validity.*

Interpretations and Satisfiability

- We use some *rules* which tell us how to obtain the meaning of an arbitrary formulae, given some interpretation.
- Before presenting these rules, we introduce a symbol: $\models$. If π is an interpretation, and ϕ is a formula, then the expression

$$\pi \models \phi$$

will be used to represent the fact that ϕ is $\top$ under the interpretation π.

Alternatively, if $\pi \models \phi$, then we say that:

  - π *satisfies* ϕ; or
  - π *models* ϕ.

- The symbol $\models$ is called the *semantic turnstile*.

- The rule for primitive propositions is quite simple. If $p \in \Phi$ then

$$\pi \models p \text{ iff } \pi(p) = T.$$

- The remaining rules are defined *recursively*.
- The rule for $\neg$:

$$\pi \models \neg\phi \text{ iff } \pi \not\models \phi$$

  (where $\not\models$ means ‘does not satisfy’.)
- The rule for $\vee$:

$$\pi \models \phi \vee \psi \text{ iff } \pi \models \phi \text{ or } \pi \models \psi$$

- Since these are the only connectives of the language, these are the only semantic rules we need.
- Since:

$$\phi \Rightarrow \psi$$

  is defined as:

$$(\neg\phi) \vee \psi$$

  it follows that:

$$\pi \models \phi \Rightarrow \psi \text{ iff } \pi \not\models \phi \text{ or } \pi \models \psi$$

- And similarly for the other connectives we defined.

- If we are given an interpretation π and a formula ϕ, it is a simple (if tedious) matter to determine whether $\pi \models \phi$.
- We just apply the rules above, which eventually bottom out of the recursion into establishing if some proposition is true or not.
- So for:

$$(p \vee q) \wedge (q \vee r)$$

  we first establish if $p \vee q$ or $q \vee r$ are true and then work up to the compound proposition.

Summary

- This lecture started to look at logic from the standpoint of artificial intelligence.
- The main use of logic from this perspective is as a means of knowledge representation.
- We introduced the basics of propositional logic.
- We also looked at a formal definition of syntax and semantics, and the properties of tautology and consistency.
- The next lecture will look at inference.